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INPUT REGULARIZATION FOR INTEGER OPTIMAL CONTROL IN BV WITH APPLICATIONS TO CONTROL OF POROELASTIC AND POROVISCOELASTIC SYSTEMS

Lorena Bociu* Paul Manns† Marvin Severitt‡ Sarah Strikwerda§

Abstract We revisit a class of integer optimal control problems for which a trust-region method has been proposed and analyzed in [40]. While the algorithm proposed in [40] successfully solves the class of optimization problems under consideration, its convergence analysis requires restrictive regularity assumptions. There are many examples of integer optimal control problems involving partial differential equations where these regularity assumptions are not satisfied. In this article we provide a way to bypass the restrictive regularity assumptions by introducing an additional partial regularization of the control inputs by means of mollification and proving a Γ-convergence-type result when the support parameter of the mollification is driven to zero. We highlight the applicability of this theory in the case of fluid flows through deformable porous media equations that arise in biomechanics. We show that the regularity assumptions are violated in the case of poroviscoelastic systems, and thus one needs to use the regularization of the control input introduced in this article. Associated numerical results show that while the homotopy can help to find better objective values and points of lower instationarity, the practical performance of the algorithm without the input regularization may be on par with the homotopy.

1 INTRODUCTION

We are interested in solving the following optimization problem

$$\text{(P)} \qquad \begin{aligned} &\min_{w\in \mathrm{BV}(0,T)} && j(w) + \alpha\,\mathrm{TV}(w) \\ &\text{s.t.} && w(t) \in W \subset \mathbb{Z} \text{ for almost all (a.a.) } t \in (0,T), \end{aligned}$$

where parameter $\alpha > 0$, time $T > 0$, $W \subset \mathbb{Z}$ is a finite set of integers, and TV denotes the total variation of the W-valued control input function w. The functional $j : L^2(0,T) \to \mathbb{R}$ is the objective that takes the form $j(w) \coloneqq J(Gw, w)$, where $J : X \to \mathbb{R}$ is a coercive and lower semicontinuous function on a Banach space X that is the state space of some partial differential equation (PDE). The function $G : L^2(0,T) \to X$ is the continuous solution operator of a PDE, which in this article will be the solution operator of a coupled system describing fluid flow through deformable porous media (see Section 5). We note that (P) admits a solution in this setting, see [39].

The optimization problem (P) falls in the class of so-called *integer optimal control problems*, which allow to model non-smooth behavior by restricting to discrete changes in distributed control variables.

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*Department of Mathematics, North Carolina State University, Raleigh, NC, USA (lvbociu@ncsu.edu)

†TU Dortmund, Dortmund, Germany (paul.manns@tu-dortmund.de)

‡Faculty of Mathematics, TU Dortmund, Dortmund, Germany (marvin.severitt@tu-dortmund.de)

§Department of Mathematics, North Carolina State University, Raleigh, NC, USA (slstrikw@ncsu.edu)

Driven by versatile applications from the optimization of supply and traffic networks [26, 27, 29, 46] over automotive control [25, 35] to topology optimization [31, 54], this problem class has attracted considerable research interest in recent years. Different methods have been proposed to treat integer optimal control problems. One of them is the combinatorial integral approximation decomposition [51] that splits the optimization into the solution of a relaxed problem, where W is replaced by a *one-hot encoding* and then the convex hull is analyzed and a fast algorithm computes a W-valued control from the relaxation [30, 43, 49, 50]. This method requires the ability to produce highly oscillating control functions, which are undesirable in many applications and can therefore not be applied to (P) if $\alpha > 0$.

The TV-term that influences (P) for $\alpha > 0$ has been prevalent in mathematical image analysis since the 1990s, see in particular the work [48]. We give the references [11, 16, 17, 23, 32, 37, 56] but note that they reflect only a small portion of the research in this area. Several authors have incorporated TV-terms in optimal control problems [12, 13, 21, 34, 41], in tight relaxations of integer optimal control problems from topology optimization [18], and in the approximation step of the combinatorial integral approximation decomposition [4, 5, 52].

A trust-region method has been proposed and analyzed to directly solve problems of the form (P) in [40]. Therein, the non-smoothness of the subproblems that arises from the distributed integer variables is handled explicitly so that the trust-region subproblems are integer linear programs after discretization. They can be solved efficiently with graph-based [53] and dynamic programming-based approaches [45]. For one-dimensional time domains $(0, T)$, like the one in (P), the TV-term of a W-valued function w is the sum of the jump heights of the function w, implying that only finitely many jumps occur because the height of a single jump is bounded below (by 1 in the case of $W \subset \mathbb{Z}$; always by some constant if $|W| < \infty$). This constitutes a desirable regularization because the application underlying (P) usually does not permit infinitely many jumps or general high-frequency switching between different control modes. This is also the case for the class of PDEs that we consider in this article to constrain our problem (P) when considering the application of tissue engineering. This is due to the fact that in the laboratory, only a finite set of values of the controls, e.g. loads that are applied in confined compression testing, are used. We highlight that although we restrict ourselves to integer-valued controls, all of the theory can be transferred straightforwardly to the case that W is a finite subset of $\mathbb{R}$ because the TV-seminorm difference between two controls is still bounded below by a positive constant if it is not zero. Therefore, we believe that these equations constitute good test cases for our algorithm.

While these optimization problems seem to be suited for the algorithmic framework proposed in [40], the regularity assumptions for its convergence analysis cannot always be satisfied. Therefore, in this article, we advance the algorithmic methodology by enforcing the necessary regularity by adding a regularization of the control input when passed to j. Specifically, the function j is altered to $j \circ K_\varepsilon$, where K_ε is a convolution operator arising from a standard mollification with parameter ε. In this article we make the following contributions in light of the algorithmic framework proposed in [40].

Contributions First we verify the regularity assumptions required in [40] for the altered objective. We prove the lower and upper bound inequalities (Γ-convergence) of the altered optimization problems when driving ε (the parameter controlling the size of the support of the mollifier) to zero. The Γ-convergence result is achieved with respect to weak* and strict convergence in the weak* closed subset of functions of bounded variations that are feasible for (P).

Consequently, global minimizers of the altered optimization problems converge to global minimizers of (P). However, the lower and upper bound inequalities do not imply that the same holds true for stationary points like the ones that are produced by the algorithm proposed in [40]. We consider a homotopy that drives $\varepsilon \to 0$ and applies meaningful termination criteria for each run (tightening of minimal trust-region radius to determine that no progress is made for $\varepsilon \to 0$ and achievement of a certain predicted reduction). We show that the cluster points are strict limits of their approximating sequences, implying that the homotopy does not *overlook* cheap reductions of the objective that can, for example, be obtained by removing small jumps from the control.

We highlight the applicability and benefits of our theory in the case of linear poroelastic and poroviscoelastic systems with incompressible constituents and distributed or boundary controls, with motivation coming from biomedicine. We show that the regularity assumptions required in [40] are satisfied in the case of poroelastic systems. As a consequence, the theory and algorithm provided in [40] can be applied. In comparison, the regularity assumptions are violated in the case of poroviscoelastic systems. Therefore, for these systems, one needs to use the regularization of the control input introduced in this article.

Lastly, we provide numerical results for two instances of the class of considered PDEs that differ in their dynamics and analytical properties. The numerical results show that while the homotopy can help to find better objective values and points of lower instationarity, the practical performance of the algorithm without the input regularization may be on par with the homotopy. Consequently, the lack of regularity may not always impair the practical performance and may therefore be outweighed by the consumption of much less running time than the homotopy.

Structure of the paper. In Section 2, we introduce our notation, briefly recall functions of bounded variation, and define the closed subset that corresponds to the feasible set of our optimization problem. In Section 3, we describe the sequential linear integer programming (SLIP) algorithm provided in [40]. We introduce and analyze the effect of the control input regularization in Section 4. The considered class of fluid-solid mixture systems and the discussion of the regularity assumptions of Algorithm 1 with respect to these coupled systems of PDEs are given in Section 5. We provide our computational setup, experiments, and results in Section 6. Finally, we draw our conclusions in Section 7.

2 NOTATION AND PRIMER ON FUNCTIONS OF BOUNDED VARIATION

Notation. Let X be a Banach space. As usual, we denote its topological dual space by the symbol X^*. For a given bounded, Lipschitz domain Ω, $L^2(\Omega)$ is the Hilbert space of square-integrable functions with inner product given by $(\cdot,\cdot)$. When the domain Ω is not clear from context, the L^2 inner product will be denoted as $(\cdot,\cdot)_\Omega$. Furthermore, we use the standard notation $H^1_{\Gamma_*}(\Omega) = \left\{ f \in H^1(\Omega) \,\middle|\, \tau f\big|_{\Gamma_*} = 0 \right\}$ where τ is the trace operator, for any $\Gamma_* \subseteq \partial\Omega$. Additionally, for any Hilbert space Y, we define the space

$$L^2(0,T;Y) = \left\{ u : [0,T] \to Y \,\middle|\, u \text{ is measurable and } \int_0^T \|u(t)\|_Y^2 dt < \infty \right\}$$

with the inner-product $(u,v)_{L^2(0,T;Y)} = \int_0^T (u(t),v(t))_Y dt$. Similarly, $H^1(0,T;Y)$ is the set of functions in $L^2(0,T;Y)$ with a time derivative in the weak sense, $u_t : [0,T] \to Y$, that belongs to $L^2(0,T;Y)$. The inner product in $H^1(0,T;Y)$ is $(u,v)_{H^1(0,T;Y)} = \int_0^T (u,v)_Y + (u_t,v_t)_Y dt$.

Functions of Bounded Variation. We give a brief summary and state the properties of functions of bounded variation, which we require in the remainder of the paper. For a detailed introduction, we refer the reader to the monograph [1]. First, we recall that a function $f : (0,T) \to \mathbb{R}$ is defined to be of bounded variation or in the space $\mathrm{BV}(0,T)$ if $f \in L^1(0,T)$ and

$$\mathrm{TV}(f) \coloneqq \sup\left\{ \int_0^T f(t)\phi'(t)\,\mathrm{d}t \,\middle|\, \phi \in C_c^1(0,T) \text{ and } \sup_{s\in(0,T)} |\phi(s)| \le 1 \right\} < \infty.$$

We recall that a sequence $(w^n)_{n\in\mathbb{N}} \subset \mathrm{BV}(0,T)$ is said to converge *weakly-** to a function $w \in \mathrm{BV}(0,T)$ if $w^n \to w$ in $L^1(0,T)$ and $\limsup_{n\to\infty} \mathrm{TV}(w^n) < \infty$. Moreover, $(w^n)_{n\in\mathbb{N}}$ is said to converge *strictly* to w if in addition $\mathrm{TV}(w^n) \to \mathrm{TV}(w)$. We define the subset of $\mathrm{BV}(0,T)$ that corresponds to the feasible set of the optimization problem (P) as

$$\mathrm{BV_W}(0,T) = \{w \in \mathrm{BV}(0,T) \mid w(t) \in W \text{ for a.a. } t \in [0,T]\}.$$

It is important for our analysis that the subset $\mathrm{BV_W}(0,T)$ is closed with respect to weak-* and strict convergence in $\mathrm{BV}(0,T)$. The closedness follows from the fact that sequences of W-valued functions that converge in $L^1(0,T)$ also have W-valued limits, which is stated explicitly for our context in [40, Lemma 2.2].

The analysis of the algorithm in [40]—the starting point of our work—makes use of regularity conditions, particularly continuity properties, that are defined for input functions in $L^2(0,T)$. While the norm-topologies of $L^1(0,T)$ and $L^2(0,T)$ are different, we note that convergence of a sequence of W-valued functions in $L^1(0,T)$ implies convergence in $L^2(0,T)$ as well (due to the fact that W is finite, implying a uniform $L^\infty(0,T)$-bound on any sequence of functions). This can be seen as follows. Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV_W}(0,T)$ be a sequence such that $u_n \overset{*}{\rightharpoonup} u$ in $\mathrm{BV_W}(0,T)$. Let w_{max} be the maximum value of W. We have

$$\begin{aligned}\|u_n - u\|^p_{L^p(0,T)} &= \int_0^T |u_n(t) - u(t)|^p dt \le \int_0^T 2^{p-1}(|u_n(t)|^{p-1} + |u(t)|^{p-1})|u_n(t) - u(t)|\, dt \\ &\le C_{p,w_{max}} \|u_n(t) - u(t)\|_{L^1(0,T)}\end{aligned}$$

Consequently, we will frequently use that sequences of functions that converge weakly-* or strictly in $\mathrm{BV_W}(0,T)$ also converge in $L^2(0,T)$.

Finally, we recall Young's inequality for convolution, see [14] p. 319, since it is used several times in the following sections: Given $f \in L^p(0,T)$ and $g \in L^q(0,T)$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \le p,q,r \le \infty$,

$$\|f * g\|_{L^r(0,T)} \le C\|f\|_{L^p(0,T)}\|g\|_{L^q(0,T)}. \tag{2.1}$$

3 SEQUENTIAL LINEAR INTEGER PROGRAMMING ALGORITHM

In order to provide a self-contained article, we provide the SLIP algorithm, which is a function space algorithm to solve (P) to stationarity [40]. Conceptually, the SLIP algorithm is a trust-region method that solves a sequence of trust-region subproblems. We briefly introduce the trust-region problem below before laying out the algorithm.

The trust-region subproblem takes a feasible control w for (P) and a trust-region radius $\Delta > 0$ as inputs and reads

$$\text{(TR)} \qquad \begin{aligned} \min_{d\in L^2(0,T)} \quad & (\nabla j(w), d)_{L^2(0,T)} + \alpha\,\mathrm{TV}(w+d) - \alpha\,\mathrm{TV}(w) =: \ell(w,d) \\ \text{s.t.} \quad & w(s) + d(s) \in W \text{ for a.a. } s \in (0,T), \\ & \|d\|_{L^1(0,T)} \le \Delta, \end{aligned}$$

where we assume that $j : L^2(0,T) \to \mathbb{R}$ is Fréchet differentiable, in particular $\nabla j(w) \in L^1(0,T)$. In section 4 we will discuss the further assumptions made in Assumption 4.1 in [40] which are required for the convergence analysis. An instance of the problem class (TR) for given w and $\Delta > 0$ is denoted by $\mathrm{TR}(w,\Delta)$ in the remainder.

We state the trust-region algorithm that solves subproblems of the form (TR) in Algorithm 1 [40]. The algorithm consists of two nested loops. In every iteration of the outer loop, which is indexed by $n \in \mathbb{N}$, the trust-region radius is reset to the input $\Delta^0 > 0$. Then the inner loop, which is indexed by $k \in \mathbb{N}$, is executed. In each inner iteration, a trust-region subproblem $\mathrm{TR}(w^{n-1}, \Delta^{n,k})$ is solved for the current trust-region radius, $\Delta^{n,k}$, and the previously accepted iterate w^{n-1} or the input w^0 (if $n - 1 = 0$). We highlight that the trust-region subproblems become integer linear programs after discretization, see [40], which can be solved to optimality with a pseudo-polynomial algorithm as detailed in [45, 53]. This is a deviation from the standard literature, where the convergence theory is developed using a Cauchy point which only guarantees a sufficient decrease and not optimality for the trust-region subproblem, see for example chapter 12 in [19]. If the predicted reduction (measured as the negative objective value of the trust-region subproblem) is zero, then the algorithm terminates. If the solution of the trust-region subproblem is acceptable (the ratio of actual reduction and predicted reduction is larger than the input $\sigma > 0$), then the inner loop terminates with new iterate w^n. If the step is rejected, then the trust region is reduced and another iteration of the inner loop is executed.

Algorithm 1 Trust-region algorithm from [40]

Input: Initial guess w^0 (feasible for (P)), $\Delta^0 > 0$, $\sigma \in (0, 1)$

```
for n = 1, ... do
    k ← 0
    Δ^{n,0} ← Δ^0
    repeat
        d^{n,k} ← minimizer of TR(w^{n-1}, Δ^{n,k})            ▷ Solve trust-region subproblem.
        if ℓ(w^{n-1}, d^{n,k}) = 0 then                        ▷ Predicted reduction is zero ⇒ terminate.
            Terminate with solution w^{n-1}.
        else if [j(w^{n-1}) + α TV(w^{n-1}) − j(w^{n-1}+d^{n,k}) − α TV(w^{n-1}+d^{n,k})] / [−ℓ(w^{n-1}, d^{n,k})] < σ then   ▷ Reject step.
            Δ^{n,k+1} ← Δ^{n,k}/2
        else                                                   ▷ Accept step.
            w^n ← w^{n-1} + d^{n,k}
        end if
        k ← k + 1
    until [j(w^{n-1}) + α TV(w^{n-1}) − j(w^{n-1}+d^{n,k-1}) − α TV(w^{n-1}+d^{n,k-1})] / [−ℓ(w^{n-1}, d^{n,k-1})] ≥ σ
end for
```

The main known convergence result on Algorithm 1 to this point is that its iterates converge to so-called L-stationary points under a suitable regularity assumption on j [40]. A feasible point w is L-stationary if the objective (P) cannot be improved further by perturbing the locations of its jumps on $(0, T)$. Such perturbations leave the term $\alpha\,\mathrm{TV}(w)$ unchanged and only affect $j(w)$, yielding a condition on $\nabla j(w)$. In particular, the condition coincides with $\nabla j(w)(t_i) = 0$ for all jump locations t_i of w if $\nabla j(w)$ is a continuous function. The formal definition is given below.

Definition 3.1. Let $v \in \mathrm{BV_W}(0, T)$ with representation $v = \chi_{(t_0,t_1)} a_1 + \sum_{i=1}^{N-1} \chi_{[t_i,t_{i+1})} a_{i+1}$ for $N \in \mathbb{N}$, $t_0 = 0$, $t_N = T$, $t_i < t_{i+1}$ for $i \in \{0, \ldots, N-1\}$, and $a_i \in W$ for $i \in \{1, \ldots, N\}$, $a_i \neq a_{i+1}$, be given. Let $j : L^2(0, T) \to \mathbb{R}$ be Fréchet differentiable, in particular $\nabla j(v) \in L^1(0, T)$. Then v is L-stationary for (P) if

1. $\overline{D_i^-}(\nabla j(v)) \geq 0 \geq \underline{D_i^+}(\nabla j(v))$ if $a_i < a_{i+1}$, and
2. $\underline{D_i^-}(\nabla j(v)) \leq 0 \leq \overline{D_i^+}(\nabla j(v))$ if $a_{i+1} < a_i$,

where

$$\overline{D_i^-}(\nabla j(v)) \coloneqq \limsup_{h\downarrow 0} \frac{1}{h}\int_{t_i-h}^{t_i} \nabla j(v)(s)\,\mathrm{d}s, \quad \underline{D_i^-}(\nabla j(v)) \coloneqq \liminf_{h\downarrow 0} \frac{1}{h}\int_{t_i-h}^{t_i} \nabla j(v)(s)\,\mathrm{d}s$$

and

$$\underline{D_i^+}(\nabla j(v)) \coloneqq \liminf_{h\downarrow 0} \frac{1}{h}\int_{t_i}^{t_i+h} \nabla j(v)(s)\,\mathrm{d}s, \quad \overline{D_i^+}(\nabla j(v)) \coloneqq \limsup_{h\downarrow 0} \frac{1}{h}\int_{t_i}^{t_i+h} \nabla j(v)(s)\,\mathrm{d}s$$

for $i \in \{1, \dots, N-1\}$.

Note that Definition 3.1 is well posed because every function in $\mathrm{BV}_{\mathrm{W}}(0,T)$ can be written in the claimed form, see, for example, [40, Proposition 4.4].

4 INPUT REGULARIZATION

This section is structured as follows. First, we recall the regularity assumptions imposed on the Hessian of the reduced objective j introduced in [40]. Under these assumptions, convergence to L-stationary points of the iterates produced by Algorithm 1 can be achieved. Secondly, motivated by our applications, we introduce weaker assumptions on the Hessian's regularity and show that the required regularity assumptions for convergence of Algorithm 1 can always be verified by regularizing (smoothing) the input of j provided that these weaker assumptions hold. Then we prove Γ-convergence in the case when the regularization is carried out by a positive mollifier and when driving the support parameter to zero. We also show that the limits of the final iterations of Algorithm 1 (under realistic termination criteria), which are in general not global minimizers but only (approximately) L-stationary points, are strict.

Assumption 4.1 (Assumption 4.1 in [40]). *Let $j : L^2(0,T) \to \mathbb{R}$ be twice Fréchet differentiable. Moreover, for all $w \in L^2(0,T)$*

$$|\nabla^2 j(w)(\psi,\phi)| \le C\|\psi\|_{L^1(0,T)}\|\phi\|_{L^1(0,T)}$$

holds for some $C > 0$ and all $\psi, \phi \in L^2(0,T)$.

Under Assumption 4.1 on the reduced objective it can be shown that the iterates produced by Algorithm 1 converge to L-stationary points [40]. We remark here that in [40] it is assumed that j is not only twice Fréchet differentiable, but that the second derivative $w \mapsto \nabla^2 j(w)$ is also continuous. This assumption enters the proofs of Lemma 4.10 and Theorem 4.23 in [40] indirectly through the employed variant of Taylor's theorem, Proposition A.1 in [40], which states that for all $u, v \in L^2(0,T)$ there exists ξ in the line segment between u and v such that $j(v) = j(u) + (\nabla j(u), v-u)_{L^2(0,T)} + \frac{1}{2}(v-u, \nabla^2 j(\xi)(v-u))_{L^2(0,T)}$ holds. Below we show that this result can be recovered in the absence of the assumption of continuity of $w \mapsto \nabla^2 j(w)$. This has been observed in [45, Section 3.2], where a slightly different formulation of the Taylor expansion is used. [1]

Proposition 4.2. *Let $j : L^2(0,T) \to \mathbb{R}$ be twice Fréchet differentiable. Let $w, \bar{w} \in L^2(0,T)$ be given. Then there exists $\xi = w + \tau(\bar{w} - w)$ for some $\tau \in [0,1]$ such that $j(\bar{w}) = j(w) + (\nabla j(w), \bar{w} - w)_{L^2(0,T)} + \frac{1}{2}(\bar{w} - w, \nabla^2 j(\xi)(\bar{w} - w))_{L^2(0,T)}$.*

Proof. We reduce the problem to the finite-dimensional case by considering the function $\tilde{j} : [0,1] \ni t \mapsto j(w + t(\bar{w} - w)) \in \mathbb{R}$. The chain rule in Banach spaces [57, Theorem 4.D] implies that $\tilde{j}''(t) = (\bar{w} - w, \nabla^2 j(w + t(\bar{w} - w))(\bar{w} - w))_{L^2(\Omega)}$ for $t \in [0,1]$. Then we apply a variant of Taylor's theorem that is

[1] The authors thank Gerd Wachsmuth for the hint to Darboux's theorem and [2].

based on Darboux's theorem and does not require continuity of the second derivative [2, Theorem 5.19], to obtain that there exists $\tau \in [0,1]$ such that $j(\bar{w}) = j(w) + (\nabla j(w), \bar{w} - w)_{L^2(\Omega)} + \frac{1}{2}(\bar{w} - w, \nabla^2 j(w + \tau(\bar{w} - w))(\bar{w} - w))_{L^2(\Omega)}$, which proves the claim with the choice $\xi \coloneqq w + \tau(\bar{w} - w)$. ◻

From our point of view, the most restrictive part of Assumption 4.1 is the boundedness of the bilinear form with respect to the product of the L^1-norms of the inputs, which is required for the convergence analysis of Algorithm 1 in [40, 44]. This is because in the setting of W-valued controls, $|W| < \infty$, the authors of [40, 44] are able to construct functions d such that in a small neighborhood $(\nabla j(w), d)_{L^2}$ decreases at least linearly with respect to $\|d\|_{L^1}$ and the $w + d$ are feasible if w is not stationary. However, $|W| < \infty$ also implies $\|d\|_{L^1} = \Theta(\|d\|_{L^2}^2)$, even $\|d\|_{L^1} = \|d\|_{L^2}^2$ if $W = \{0,1\}$. Therefore, in order to dominate the quadratic term in the proofs, the boundedness with respect to the product of the L^1-norms is assumed, see also the related comments in [40, 44]. A related assumption (Lipschitz continuity of the derivative) is made in (5), (10) in [28] (note that the abstract space Y therein becomes L^1 for the examples). However, Assumption 4.1 may be considered to be too restrictive for practical applications of Algorithm 1 because it requires an improvement of the input regularity of the control-to-state operator. In particular, as we also experience for the considered poroviscoelastic problem in Section 5, one may only be able to verify one of the weaker assumptions below, where the uniform boundedness of the Hessian of j is assumed with respect to stronger norms for the control input.

Assumption 4.3. *Let $j : L^2(0,T) \to \mathbb{R}$ be twice Fréchet differentiable such that for all $w \in L^2(0,T)$*

$$|\nabla^2 j(w)(\psi, \phi)| \le C\|\psi\|_{L^2(0,T)}\|\phi\|_{L^2(0,T)}$$

holds for some $C > 0$ and all $\psi, \phi \in L^2(0,T)$.

Assumption 4.4. *Let $j : H^1(0,T) \to \mathbb{R}$ be twice Fréchet differentiable such that for all $w \in H^1(0,T)$*

$$|\nabla^2 j(w)(\psi, \phi)| \le C\|\psi\|_{H^1(0,T)}\|\phi\|_{H^1(0,T)}$$

holds for some $C > 0$ and all $\psi, \phi \in H^1(0,T)$.

Regularization of control inputs. Let j_ε be defined as follows:

$$j_\varepsilon \coloneqq j \circ K_\varepsilon, \tag{4.1}$$

where $K_\varepsilon : L^1(0,T) \to L^2(0,T)$ is a (bounded and linear) convolution operator defined as

$$K_\varepsilon(w) \coloneqq r_{[0,T]}(\eta_\varepsilon * w), \text{ for any } w \in L^1(0,T), \tag{4.2}$$

where $(\eta_\varepsilon)_{\varepsilon>0}$ is a family of positive mollifiers [42, Section 4.5], and $r_{[0,T]}$ denotes the restriction of a function defined on all of $\mathbb{R}$ to the interval $[0,T]$. For the sake of the convolution being well-defined we assume that all $w \in L^1(0,T)$ are extended to 0 outside of $[0,T]$ when passed into the convolution operation.

Then the following three propositions hold.

Proposition 4.5. *Let $\varepsilon > 0$. Then $K_\varepsilon : L^1(0,T) \to L^2(0,T)$ defined above in (4.2) is a bounded linear operator. If j satisfies Assumption 4.3, then j_ε satisfies Assumption 4.1 with some $C = C_\varepsilon$ that depends on ε.*

Proof. Let $w \in L^1(0,T)$. Let $K_\varepsilon^* : L^2(0,T) \to L^\infty(0,T)$ denote the adjoint operator of the bounded linear operator K_ε, where we have identified $(L^2(0,T))^* \cong L^2(0,T)$ and $(L^1(0,T))^* \cong L^\infty(0,T)$. The chain rule, see, [57, Theorem 4.D], yields the derivatives

$$\nabla j_\varepsilon(w) = K_\varepsilon^* \nabla j(K_\varepsilon w) \quad \text{and} \quad \nabla^2 j_\varepsilon(w) = \langle K_\varepsilon^* \nabla^2 j(K_\varepsilon w) K_\varepsilon \cdot, \cdot \rangle_{L^\infty(0,T), L^1(0,T)}.$$

Let $\phi, \psi \in L^1(0,T)$. Cauchy–Schwarz inequality and the submultiplicativity of the operator norm give

$$\langle K_\varepsilon^* \nabla^2 j(K_\varepsilon w) K_\varepsilon \psi, \phi \rangle_{L^\infty(0,T),L^1(0,T)} = (\nabla^2 j(K_\varepsilon w) K_\varepsilon \psi, K_\varepsilon \phi)_{L^2(0,T)} \le C \|K_\varepsilon \psi\|_{L^2(0,T)} \|K_\varepsilon \phi\|_{L^2(0,T)}$$

where $C > 0$ is the constant from Assumption 4.3. By virtue of Young's convolution inequality, we obtain $\|K_\varepsilon \phi\|_{L^2(0,T)} \le \|\eta_\varepsilon\|_{L^2(\mathbb{R})} \|\phi\|_{L^1(0,T)}$. Because $(\eta_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers we have that $\|\eta_\varepsilon\|_{L^2(\mathbb{R})} < \infty$ for all $\varepsilon > 0$, and thus the claims follow. □

Proposition 4.6. *Let $\varepsilon > 0$. Then $K_\varepsilon : L^1(0,T) \to H^1(0,T)$ defined above in (4.2) is a bounded linear operator. If j satisfies Assumption 4.4, then j_ε satisfies Assumption 4.1 with some $C = C_\varepsilon$ that depends on ε.*

Proof. The proof is very similar to that of Proposition 4.5 above. We need to show that

$$\|K_\varepsilon \phi\|_{H^1} = \left(\|K_\varepsilon \phi\|_{L^2}^2 + \left\| \frac{\mathrm{d}}{\mathrm{d}t} K_\varepsilon \phi \right\|_{L^2}^2 \right)^{\frac{1}{2}} \le C \|\phi\|_{L^1},$$

for some $C > 0$. From Proposition 4.5 we already have that $\|K_\varepsilon \phi\|_{L^2} \le c_1 \|\phi\|_{L^1}$. Now using the formula for the derivative of a convolution $\frac{\mathrm{d}}{\mathrm{d}t}(\eta_\varepsilon * \phi) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta_\varepsilon\right) * \phi$ and Young's convolution inequality, we obtain that $\left\|\frac{\mathrm{d}}{\mathrm{d}t} K_\varepsilon \phi\right\|_{L^2} \le c_2 \|\phi\|_{L^1}$ holds with $c_2 := \left\|\frac{\mathrm{d}}{\mathrm{d}t}\eta_\varepsilon\right\|_{L^2}$, which is bounded because η_ε is smooth. □

Proposition 4.7. *Let $(\eta_\varepsilon)_{\varepsilon>0}$ be a family of standard mollifiers. Let j_ε be defined as above in (4.1). Let $w \in L^2(0,T)$. Then $\nabla j_\varepsilon(w) = K_\varepsilon^* \nabla j(K_\varepsilon w) \in L^\infty(0,T)$ is a continuous function on $[0,T]$.*

Proof. As above, the identification $L^\infty(0,T) \cong (L^1(0,T))^*$ implies that we can consider $K_\varepsilon^* \nabla j(K_\varepsilon w)$ as an $L^\infty(0,T)$-function. Moreover, $\nabla j(K_\varepsilon w)$ is an $L^2(0,T)$-function. Let $f \in L^1(0,T)$, $g \in L^2(0,T)$. We consider the adjoint operator K_ε^*, which is defined by the identity

$$(K_\varepsilon f, g)_{L^2(0,T)} = \langle f, K_\varepsilon^* g \rangle_{L^1(0,T),L^\infty(0,T)}.$$

We insert the definition of the convolution and obtain the following identity

$$\int_0^T \overbrace{\int_0^T \eta_\varepsilon(t-s) f(s)\,\mathrm{d}s}^{(K_\varepsilon f)(t)}\, g(t)\,\mathrm{d}t = \int_0^T f(s) \overbrace{\int_0^T \eta_\varepsilon(t-s) g(t)\,\mathrm{d}t}^{(K_\varepsilon^* g)(s)}\,\mathrm{d}s$$

by virtue of Fubini's theorem. $K_\varepsilon f \in C^\infty([0,T])$ holds as $\eta_\varepsilon * f$ is the convolution of f and a mollifier, see Theorem 7 in [22]. Due to the structural symmetry of the mollification (or also of the mollifiers themselves), the proof for smoothness of $K_\varepsilon f$ on $[0,T]$ can be transferred directly to $K_\varepsilon^* g$ on $[0,T]$. □

Let (P_ε) be the optimization problem (P) where the objective j is replaced by j_ε. We note that the constant C_ε asserted by Propositions 4.5 and 4.6 blows up for $\varepsilon \searrow 0$ so that the property can not be carried over to the limit. We are interested in the ability of local solutions, global solutions, and stationary points of the new optimization problem (P_ε) to approximate local solutions, global solutions, and stationary points of (P) as $\varepsilon \to 0$. While we are not able to give a full answer to the question at this point, we can provide a positive answer in the case of global solutions by virtue of Γ-convergence. Assume that the objectives $(j_\varepsilon + \alpha\,\mathrm{TV})$ Γ-converge to $(j + \alpha\,\mathrm{TV})$ as $\varepsilon \to 0$ with respect to $\mathrm{BV}(0,T)$-weak-*-convergence on the complete subspace $\mathrm{BV}_W(0,T)$, which is the feasible set of (P). Then we obtain that global minimizers of j_ε in $\mathrm{BV}_W(0,T)$ weakly-*-converge to global minimizers of j in $\mathrm{BV}_W(0,T)$, which is one of our main results.

Theorem 4.8. *Let j be continuous. For $\varepsilon > 0$, let $(\eta_\varepsilon)_{\varepsilon>0}$ be a family of positive mollifiers. Let K_ε be defined as above in (4.2).*

Then the sequence $(\hat{j}_{\varepsilon^n})_{n\in\mathbb{N}}$, defined as $\hat{j}_{\varepsilon^n} := j_{\varepsilon^n} + \alpha\,\mathrm{TV}$, Γ-converges to $\hat{j} = j + \alpha\,\mathrm{TV}$ as $\varepsilon^n \to 0$ with respect to weak--convergence and strict convergence in $\mathrm{BV_W}(0,T)$.*

Proof. We start by proving the lower bound inequality. To this end, let $w^{\varepsilon^n} \overset{*}{\rightharpoonup} w$ in $\mathrm{BV_W}(0,T)$. Then $\mathrm{TV}(w) \leq \liminf_{\varepsilon^n\to 0} \mathrm{TV}(w^{\varepsilon^n})$ by virtue of the weak-*-lower semicontinuity of the total variation.

We now show that $j_{\varepsilon^n}(w^{\varepsilon^n}) \to j(w)$. Due to the continuity of j it suffices to show that

$$\|K_{\varepsilon^n} w^{\varepsilon^n} - w\|_{L^2(0,T)} \to 0.$$

We observe that

$$\|K_{\varepsilon^n} w^{\varepsilon^n} - w\|_{L^2(0,T)} \leq \|K_{\varepsilon^n} w^{\varepsilon^n} - K_{\varepsilon^n} w\|_{L^2(0,T)} + \|K_{\varepsilon^n} w - w\|_{L^2(0,T)}.$$

Using Young's convolution inequality we obtain

$$\|K_{\varepsilon^n} w^{\varepsilon^n} - K_{\varepsilon^n} w\|_{L^2(0,T)} = \|K_{\varepsilon^n}(w^{\varepsilon^n} - w)\|_{L^2(0,T)} \leq \|\eta_{\varepsilon^n}\|_{L^1(\mathbb{R})} \|w^{\varepsilon^n} - w\|_{L^2(0,T)} \leq \|w^{\varepsilon^n} - w\|_{L^2(0,T)} \to 0.$$

The convergence $\|K_{\varepsilon^n} w - w\|_{L^2(0,T)} \to 0$ follows from Theorem 7, page 714, in [22].

To prove the upper bound inequality, we choose $w^{\varepsilon^n} := w$ for all $n \in \mathbb{N}$ and observe

$$\hat{j}_{\varepsilon^n}(w^{\varepsilon^n}) = j_{\varepsilon^n}(w) + \alpha\,\mathrm{TV}(w) \to j(w) + \alpha\,\mathrm{TV}(w)$$

with the same argument as above. This proves Γ-convergence with respect to weak-*-convergence in $\mathrm{BV_W}(0,T)$. Because the chosen sequence for the upper bound inequality is also strictly convergent, Γ-convergence also holds with respect to strict convergence in $\mathrm{BV_W}(0,T)$. □

Proposition 4.5 shows it is sufficient to verify the much weaker assumptions Assumption 4.3 or Assumption 4.4 instead of Assumption 4.1 when we solve (P) with the input regularization j_ε instead of j for some $\varepsilon > 0$. Proposition 4.7 implies that $\nabla j_\varepsilon(w)$, with $w \in \mathrm{BV_W}(0,T)$, is a smooth function on the interval $[0,T]$. Thus we can use the characterization

$$\nabla j_\varepsilon(w)(t_i) = 0$$

for all of the finitely many t_i where the function w has a jump, if w is L-stationary for (P) with j_ε instead of j, see [40].

While we obtain convergence of global minimizers under Assumption 4.3, we do not know at present if we obtain converge to an L-stationary point of (P) when we compute L-stationary points for (P) with j_ε instead of j and drive ε to zero. In particular, we have not been able to show that the limit of L-stationary or approximately L-stationary points of (P) with j_ε that are produced by Algorithm 1 for a homotopy that drives $\varepsilon \to 0$ is L-stationary for (P). We note that even if this were true, L-stationary is not known to be a necessary optimality condition for (P) if only Assumption 4.3 but not Assumption 4.1 holds.

Moreover, the situation is even worse in case that we only have Assumption 4.4. Because $H^1(0,T) \hookrightarrow C([0,T])$, the only W-valued functions in $H^1(0,T)$ are constant on the whole the domain $(0,T)$, so we cannot prove a result like Theorem 4.8 in this case and the limit problem $\varepsilon = 0$ has no meaningful interpretation due to the high regularity that is required for the control input. However, in case one is still interested in discrete-valued controls even if this is not covered by the regularity theory for the PDE, we believe that solving (P) with j_ε instead of j is still sensible because for a given control $w \in \mathrm{BV_W}(0,T)$ we know that there are only finitely many switches (or jumps) that can occur and by means of the parameter ε we can control the support size of the smooth transitions between them that occurs when mollifying them.

However, the succeeding analysis shows that in case of Assumption 4.3, weak-* limit points of the homotopy are also strict limit points, which means that the limit cannot have a lower total variation than its approximating sequence. Consequently, there is no nearby reduction of the objective function by an improvement of the TV-term. This result is possible even in the presence of realistic early termination criteria in Algorithm 1.

Proposition 4.9. *Let j be bounded below. Assume that Algorithm 1 is terminated when one of the following conditions is met:*

- *the trust-region radius is smaller than a given $C_1 > 0$*
- *the predicted reduction $-\ell(w^{n-1}, d^{n,k})$ in outer iteration $n \in \mathbb{N}$ and inner iteration $k \in \mathbb{N}$ is smaller than a given $C_2 > 0$,*

then it terminates within finitely many outer iterations regardless of the initial guess w^0.

Remark 4.10. Before proving Proposition 4.9, we make a brief note to explicitly explain how these conditions enter Algorithm 1. The first condition is checked after each reduction of the trust-region radius together with Algorithm 1 ln. 14. The second condition replaces the termination criterion in Algorithm 1 ln. 6.

Proof of Proposition 4.9. The first condition ensures that the inner loop iterates only finitely many times, specifically at most $\lfloor \log_2(C_1/\Delta^0) \rfloor$ times. Assume that Algorithm 1 does not terminate within finitely many iterations of the outer loop. Then the inner loop accepts a new iterate within $k \leq \lfloor \log_2(C_1/\Delta^0) \rfloor$ iterations in each outer iteration $n \in \mathbb{N}$. The step acceptance implies that the reduction in the objective is always higher than $-\sigma\ell(w^{n-1}, d^{n,k})$. It follows from the second termination criterion that $-\ell(w^{n-1}, d^{n,k}) \geq C_2$ for all $n \in \mathbb{N}$ and corresponding k on acceptance. This contradicts that j and TV are bounded from below. Consequently, Algorithm 1 terminates within finitely many outer iterations. □

Proposition 4.11. *Let j be bounded below. Let Assumption 4.3 hold. Let $(w_\varepsilon)_\varepsilon \subset \mathrm{BV_W}(0,T)$ be the sequence of final iterates produced by Algorithm 1 executed on j_ε for a sequence $\varepsilon \to 0$ with the initial control given by the previous final iterate, where the execution of Algorithm 1 is terminated if one of following conditions is met:*

- *the trust region radius is smaller than $\Delta(\varepsilon) > 0$, which tends to zero as ε is driven to zero,*
- *the predicted reduction is smaller than $(1 - \frac{3a}{4})\alpha > 0$ for some fixed $0 < a \leq 1 - \sigma < 1$.*

If $\liminf_{\varepsilon \to 0} \mathrm{TV}(w_\varepsilon) < \infty$, there is at least one weakly- convergent subsequence. Every weakly-* convergent subsequence converges strictly to a limit point in $\mathrm{BV_W}(0,T)$.*

Proof. First, we note that the sequence $(w_\varepsilon)_\varepsilon$ is well defined because Proposition 4.9 asserts that Algorithm 1 terminates after finitely many iterations for the two assumed termination criteria. Second, the facts that $\liminf_{\varepsilon \to 0} \mathrm{TV}(w_\varepsilon) < \infty$ and that $(w_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T)$ and hence $L^1(0,T)$ imply that there is at least one weakly-* convergent subsequence.

In the remainder of the proof we consider an arbitrary weak-* convergent subsequence $w_i \stackrel{*}{\rightharpoonup} \bar{w}$ for $\varepsilon_i \to 0$ in $\mathrm{BV}(0,T)$. The convergence theory of Algorithm 1 in [40] gives $(w_\varepsilon)_\varepsilon \subset \mathrm{BV_W}(0,T)$ and the weak-* closedness of $\mathrm{BV_W}(0,T)$ (see Section 2) gives $\bar{w} \in \mathrm{BV_W}(0,T)$. We note that the subsequence also converges in $L^2(0,T)$ by boundedness in $L^\infty(0,T)$ (due to $w_i(t) \in W$ a.e. for all ε_i) and pointwise convergence a.e. of a subsequence.

We set forth to prove the claim by way of contradiction. To this end, assume that the convergence of the subsequence is not strict. Then we can find a subsequence, for ease of notation also denoted by w_i, and $\bar{C} > 0$ such that

$$\alpha \operatorname{TV}(w_i) - \alpha \operatorname{TV}(\bar{w}) \geq \bar{C} \geq \alpha > 0 \tag{4.3}$$

for all $i \in \mathbb{N}$ because the TV-seminorm is weak-* lower semi-continuous. Because $\operatorname{TV}(w_i), \operatorname{TV}(\bar{w}) \in \mathbb{Z}$, we can assume that $\bar{C} \geq \alpha$ holds. Note

$$|j(K_{\varepsilon_i} w_i) - j(\bar{w})| \leq |j(K_{\varepsilon_i} w_i) - j(K_{\varepsilon_i}\bar{w})| + |j(K_{\varepsilon_i}\bar{w}) - j(\bar{w})|.$$

Additionally, $\|K_{\varepsilon_i}\bar{w} - \bar{w}\|_{L^2} \to 0$ holds by the argumentation provided in Theorem 7, page 714, in [22]. Young's convolution inequality shows

$$\|K_{\varepsilon_i} w_i - K_{\varepsilon_i}\bar{w}\|_{L^2} \leq \|\eta_{\varepsilon_i}\|_{L^1}\|w_i - \bar{w}\|_{L^2} = \|w_i - \bar{w}\|_{L^2} \to 0.$$

Thus, we obtain that $j(K_{\varepsilon_i} w_i) \to j(\bar{w})$ for $i \to \infty$.

Let $\tilde{\Delta} > 0$ be arbitrary but fixed. Then there is $i_0 \in \mathbb{N}$ large enough such that for all $i \geq i_0$

$$\|w_i - \bar{w}\|_{L^1} \leq \tilde{\Delta} \quad \text{and} \quad \|\nabla j(w_i) - \nabla j(\bar{w})\|_{L^2} \leq \tilde{\Delta}$$

hold. Thus $\bar{w} - w_i$ is feasible for $\operatorname{TR}(w_i, \tilde{\Delta})$ for all $i \geq i_0$. Let $w_0 \in \operatorname{BV}_{\mathrm{W}}(0, T)$ be an arbitrary point such that $w_0 - w_i$ is feasible for $\operatorname{TR}(w_i, \tilde{\Delta})$. Using Taylor's theorem we obtain that

$$\begin{aligned} j(K_{\varepsilon_i} w_i) - j(K_{\varepsilon_i} w_0) = {} & \sigma(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w_0))_{L^2} + (1-\sigma)(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w_0))_{L^2} \\ & + \frac{1}{2}\nabla^2 j(\xi_i)(K_{\varepsilon_i}(w_i - w_0), K_{\varepsilon_i}(w_i - w_0)) \end{aligned}$$

for some $\xi_i \in L^2(0, T)$. From Assumption 4.3 we derive that

$$|\nabla^2 j(\xi_i)(K_{\varepsilon_i}(w_i - w_0), K_{\varepsilon_i}(w_i - w_0))| \leq C\|w_i - w_0\|_{L^2}^2 \leq C|\max(W) - \min(W)|\|w_i - w_0\|_{L^1} \leq C_2\tilde{\Delta}$$

for some $C, C_2 > 0$. Furthermore, we obtain that

$$\begin{aligned} |(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w_0))_{L^2}| &\leq \|\nabla j(K_{\varepsilon_i} w_i)\|_{L^2}\|K_{\varepsilon_i}(w_i - w_0)\|_{L^2} \\ &\leq (\|\nabla j(\bar{w})\|_{L^2} + C_3\tilde{\Delta})\|w_i - w_0\|_{L^2} \\ &\leq C_4(C_5 + C_3\tilde{\Delta})\sqrt{\tilde{\Delta}} \end{aligned}$$

holds for some $C_3, C_4, C_5 > 0$. Then there exists $\tilde{\Delta}$ as above such that for i_0 corresponding to $\tilde{\Delta}$ we obtain for all $i \geq i_0$ that the estimates

$$\Delta(\varepsilon_i) < \frac{\tilde{\Delta}}{2}, \tag{4.4}$$

$$|\nabla^2 j(\xi_i)(K_{\varepsilon_i}(w_i - w_0), K_{\varepsilon_i}(w_i - w_0))| \leq \frac{a}{4}\bar{C} \text{ and} \tag{4.5}$$

$$|(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w_0))_{L^2}| \leq \frac{a}{4}\bar{C} \tag{4.6}$$

hold for all w_0 in the trust region of $\operatorname{TR}(w_i, \Delta)$ if $\Delta \leq \tilde{\Delta}$. Moreover, there exists a minimal $\tilde{k} \in \mathbb{N}$ such that $\tilde{\Delta} \geq \Delta^0 2^{-\tilde{k}}$.

We now show that the inner loop of Algorithm 1 accepts the step not later than in inner iteration $\tilde{k}$

for i large enough. Specifically, for all $i \geq i_0$, the execution of Algorithm 1 for ε_i accepts a step not later than in inner iteration $\tilde{k}$ in the final outer iteration. The trust-region radius upon acceptance is $\Delta^0 2^{-\tilde{k}}$ with $\Delta(\varepsilon_i) < \Delta^0 2^{-\tilde{k}} \leq \tilde{\Delta} \leq \Delta^0 2^{-\tilde{k}+1}$.

Let $w^* - w_i$ be the optimal solution of $\mathrm{TR}(w_i, \Delta^0 2^{-\tilde{k}})$. Because $\bar{w} - w_i$ is in the trust region of $\mathrm{TR}(w_i, \Delta^0 2^{-\tilde{k}})$ it follows that

$$\alpha(\mathrm{TV}(w_i) - \mathrm{TV}(w^*)) \geq \alpha(\mathrm{TV}(w_i) - \mathrm{TV}(\bar{w})) - \frac{a}{2}\bar{C} \geq \left(1 - \frac{a}{2}\right)\bar{C} \tag{4.7}$$

because the predicted reduction of w^* is greater than or equal to that of $\bar{w}$. Let $M \coloneqq \alpha(\mathrm{TV}(w_i) - \mathrm{TV}(w^*)) - (1 - \frac{a}{2})\bar{C} \geq 0$. In total we obtain the inequalities

$$\begin{aligned}
&j(K_{\varepsilon_i} w_i) - j(K_{\varepsilon_i} w^*) + \alpha(\mathrm{TV}(w_i) - \mathrm{TV}(w^*)) \\
&\quad = \sigma(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w^*))_{L^2} + (1-\sigma)(\nabla j(K_{\varepsilon_i} w_i), K_{\varepsilon_i}(w_i - w^*))_{L^2} \\
&\quad\quad + \frac{1}{2}\nabla^2 j(\xi)(K_{\varepsilon_i}(w_i - w^*), K_{\varepsilon_i}(w_i - w^*)) + \alpha(\mathrm{TV}(w_i) - \mathrm{TV}(w^*)) \\
&\quad \geq -\sigma\ell(w_i, w^* - w_i) - \frac{a}{4}\bar{C} - \frac{a}{4}\bar{C} + (1-\sigma)\left(1 - \frac{a}{2}\right)\bar{C} + (1-\sigma)M && \text{(4.5),(4.6)} \\
&\quad \geq -\sigma\ell(w_i, w^* - w_i) - \frac{a}{4}\bar{C} - \frac{a}{4}\bar{C} + \frac{a}{2}\bar{C} + (1-\sigma)M && a \leq 1 - \sigma \\
&\quad \geq -\sigma\ell(w_i, w^* - w_i) \\
&\quad \geq \sigma\left(1 - \frac{3a}{4}\right)\bar{C} \underset{a<1}{>} 0. && \text{(4.6),(4.7)}
\end{aligned}$$

We analyze these inequalities with respect to the two possible termination criteria that are assumed (note that the original termination criterion in Algorithm 1 ln. 6 is replaced by the second one, see also Remark 4.10).

The first termination criterion does not apply, because the trust-region algorithm would find an improvement of $\sigma(1 - \frac{3a}{4})\bar{C}$ before the critical trust-region radius is attained. The second termination criterion does not apply, because the predicted reduction would have to be less than $(1 - \frac{3a}{4})\alpha \leq (1 - \frac{3a}{4})\bar{C}$ due to inequality 4.3, which can not happen as the inequalities show.

Thus neither of the assumed termination criteria is satisfied for any $i \geq i_0$ but the w_i are final iterates of Algorithm 1 under the assumed termination criteria, which is a contradiction. Consequently, the convergence of w_i to $\bar{w}$ is strict. ◻

Remark 4.12. We note that the proof does not require the assumption that an execution of Algorithm 1 is initialized with the final iterate of the execution for the previous choice of ε. We have included it here because it is the natural choice for a homotopy that drives ε to zero and generally helps to obtain a bounded (and improving) sequence in practice as can also be seen in our numerical examples.

We note that a there are ample ways to explore more sophisticated and more efficient homotopy methods with adaptive choices of ε^n so that the homotopy becomes integrated into Algorithm 1.

We believe that if the trust-region subproblems (TR) are solved inexactly, the convergence analysis in Proposition 4.11 as well as in [40] can still be carried out if one can guarantee that the inexact solution to (TR) has an objective value that is smaller than optimal objective value multiplied by a fixed constant $c \in (0, 1)$. Clearly, c will appear in the constants in the arguments and statements in this case. Moreover, we believe this can be further combined with inexact evaluations of ∇j_ε when this inexactness is driven to zero over the course of the iterations.

5 Application to fluid flows through deformable, porous media

PDE Model. Let $\Omega \subset \mathbb{R}^3$ be the open, bounded domain occupied by the fluid-solid mixture, with Lipschitz boundary $\partial\Omega$. Motivated by applications in biomechanics (like tissue perfusion [3, 15, 24, 36, 38, 47]), we work under the assumptions of full saturation, negligible inertia, small deformations and incompressible mixture components (in the sense that the solid and fluid phases can't undergo volume changes at the microscale). Due to the complex composition of biological tissue, which exhibit both elastic and viscoelastic behaviors, we consider both poroelastic and poroviscoelastic systems, where the effective stress tensor is of Kelvin-Voigt type. The extent to which structural viscoelasticity is present in the equations is represented by the parameter $\delta \geq 0$. Therefore, the total stress of the fluid-solid mixture is given by

$$\mathbf{T}(\mathbf{u}, p) = \delta\mu_v(\nabla\mathbf{u}_t + \nabla\mathbf{u}_t^T) + \delta\lambda_v(\nabla\cdot\mathbf{u}_t)\mathbf{I} + \mu_e(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda_e(\nabla\cdot\mathbf{u})\,\mathbf{I} - p\mathbf{I},$$

where $\mathbf{u}$ is the elastic displacement and p is the fluid pressure. Moreover, $\mathbf{I}$ stands for the identity tensor, λ_e and μ_e are the Lamé parameters, and λ_v, μ_v are the visco-elastic parameters, which are all strictly positive.

The quasi-static system is described by two conservation laws: the balance of linear momentum for the fluid-solid mixture and the balance of mass for the fluid component.

$$\nabla\cdot\mathbf{T}(\mathbf{u}, p) + \mathbf{F}(\mathbf{x}, t) = \mathbf{0} \quad \text{and} \quad \zeta_t + \nabla\cdot\mathbf{v} = S(\mathbf{x}, t) \quad \text{in } \Omega\times(0, T), \tag{5.1}$$

where the fluid content ζ is given by $\zeta = \nabla\cdot\mathbf{u}$, and the discharge velocity $\mathbf{v}$ is given by $\mathbf{v} = -\mathbf{K}\nabla p$, with $\mathbf{K} = k\mathbf{I}$, where $\mathbf{K}$ is the permeability tensor and k is a constant.We note here that the formula for the fluid content is a simplification of the more general expression $\zeta = c_0 p + \alpha\nabla\cdot\mathbf{u}$ [6], where c_0 is the constrained specific storage coefficient and α is the Biot-Willis coefficient. The simplification is made due to the assumption of incompressible fluid and solid components of the mixture (biological tissues have a mass density close to that of the water), which mathematically translates to $c_0 = 0$ and $\alpha = 1$ [20], and therefore the fluid content becomes solid dilation.

We assume that $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are the Dirichlet and Neumann parts of the boundary (with respect to the elastic displacement), with $\Gamma_D \cap \Gamma_N = \emptyset$ (while allowing $\overline{\Gamma}_D \cap \overline{\Gamma}_N \neq \emptyset$).

We associate the following boundary and initial conditions to the balance laws mentioned above:

$$\mathbf{T}(\mathbf{u}, p)\mathbf{n} = \mathbf{g}, \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N\,, \tag{5.2}$$

$$\mathbf{u} = \mathbf{0}, \quad p = 0 \quad \text{on } \Gamma_{D,p}\,, \tag{5.3}$$

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{v}\cdot\mathbf{n} = \psi \quad \text{on } \Gamma_{D,\mathbf{v}}\,. \tag{5.4}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \text{ in } \Omega. \tag{5.5}$$

Note that the Dirichlet part of the boundary $\Gamma_D = \Gamma_{D,p} \cup \Gamma_{D,\mathbf{v}}$, where the subscripts p and $\mathbf{v}$ indicate conditions imposed on the Darcy pressure and discharge velocity, respectively. As usual, $\mathbf{n}$ is the outward unit normal vector.

The data in the system is represented by the body force per unit of volume $\mathbf{F}$, the net volumetric fluid production rate S, and the boundary sources $\mathbf{g}$ and ψ. They can be used as controls.

We impose the following assumptions on the domain:

Assumption 5.1. *We assume:*

1. Γ_D *is a set of positive measure, so by Korn's inequality:*

$$E(\mathbf{u}(t)) = \|\nabla \cdot \mathbf{u}\|^2_{L^2(\Omega)} + (\nabla \mathbf{u} : (\nabla \mathbf{u} + \nabla \mathbf{u}^T)) \geq c||\mathbf{u}(t)||^2_{\mathbf{H}^1(\Omega)}$$

where $\nabla \mathbf{u}$ *stands for the Jacobian matrix of* $\mathbf{u}$ *and the Frobenius inner product of two matrices is given by*

$$(\mathbf{A} : \mathbf{B}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega (A_{ij} B_{ij}) d\Omega.$$

2. $\Gamma_{D,p}$ *is a set of positive measure, so by Poincare's inequality:*

$$||v||_{L^2(\Omega)} \leq C_P ||\nabla v||_{\mathbf{L}^2(\Omega)}, \quad \forall v \in V.$$

Notation. Let

$$\mathbb{V} \equiv \mathbf{V} \times V = (H^1_{\Gamma_D}(\Omega))^3 \times H^1_{\Gamma_{D,p}}(\Omega),$$

where the inner-product on $\mathbf{V}$ is given by

$$a(\mathbf{u}, \mathbf{w}) = (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{w}) + (\nabla \mathbf{u} : \nabla \mathbf{w}) + (\nabla \mathbf{u} : (\nabla \mathbf{w})^T). \tag{5.6}$$

We note that the bilinear form $a(\cdot,\cdot)$ defines an inner product on $\mathbf{V}$, due to Assumption 5.1 on the domain. The inner product for V is inherited from $H^1(\Omega)$.

We have the following results on well-posedness of weak solutions [8–10]:

Proposition 5.2 (Poroelasticity). *Let* $\delta = 0, \mathbf{F} \in H^1(0,T;\mathbf{L}^2(\Omega)), S \in L^2(0,T;L^2(\Omega))$*, and* $\mathbf{g} \in H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))$*. Additionally, let* $\psi(x,t) = w(t)\chi(x)$ *where* $w(t) \in L^2(0,T)$ *and* $\chi \in L^2(\Gamma_{D,v})$*. Then there exists a unique weak solution* $(\mathbf{u}, p) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;V)$ *to (5.1-5.5). Additionally, the solution satisfies the following energy estimate:*

$$\int_0^T \|p\|^2_V + \|\mathbf{u}\|^2_{\mathbf{V}} dt \leq C(\|\mathbf{u}_0\|_{\mathbf{V}} + \|\mathbf{F}\|^2_{H^1(0,T;\mathbf{L}^2(\Omega))} + \|S\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{g}\|^2_{H^1(0,T;\mathbf{H}^{1/2}(\Gamma_N))} + \|\psi\|^2_{L^2(0,T;L^2(\Gamma_{D,v}))}) \tag{5.7}$$

Proposition 5.3 (Poroviscoelasticity). *Let* $\delta > 0$, $\mathbf{F} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $S \in L^2(0,T;L^2(\Omega))$*, and* $\mathbf{g} \in L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))$*. Additionally, let* $\psi(x,t) = w(t)\chi(x)$ *where* $w(t) \in L^2(0,T)$ *and* $\chi \in L^2(\Gamma_{D,v})$*. Then there exists a unique weak solution* $(\mathbf{u}, p) \in H^1(0,T;\mathbf{V}) \times L^2(0,T;V)$ *to (5.1-5.5). Additionally, the solution satisfies the following energy estimate:*

$$\begin{aligned} &\int_0^T \|p\|^2_V + \|\mathbf{u}\|^2_{\mathbf{V}} + \delta\|\mathbf{u}_t\|^2_{\mathbf{V}} dt \\ &\leq C(\|\mathbf{u}_0\|_{\mathbf{V}} + \|\mathbf{F}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|S\|^2_{L^2(0,T;L^2(\Omega))} + \|\mathbf{g}\|^2_{L^2(0,T;\mathbf{H}^{1/2}(\Gamma_N))} + \|\psi\|^2_{L^2(0,T;L^2(\Gamma_{D,v}))}) \end{aligned} \tag{5.8}$$

Similar results hold when the control w is used as the time portion of S. Therefore, we will let $\chi(x)$ be fixed and insert $q = \chi(x)w(t)$ into (5.1)-(5.5) in place of either S or ψ. Let $G : L^2(0,T) \to L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Omega))$ be defined by mapping $w(t) \in L^2(0,T)$ to the unique solution of (5.1)-(5.5), embedded in $L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Omega))$, with all sources set to zero except for q. Let $(\tilde{\mathbf{u}}, \tilde{p})$ be the unique solution (in the spaces provided in Proposition 5.2 and Proposition 5.3) to (5.1)-(5.5) with the sources set as desired and $q = 0$. We introduce the control problem

$$
(5.9)\qquad \begin{aligned} \min_{w\in L^2(0,T)} \quad & \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2}\|p-p_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\|w\|^2_{L^2(0,T)} + \alpha\,\mathrm{TV}(w) \\ \text{s.t.} \quad & (\mathbf{u},p) = Gw + (\tilde{\mathbf{u}},\tilde{p}), \\ & w(t)\in W \text{ for a.a. } t\in(0,T) \end{aligned}
$$

for given $\mathbf{u}_d \in L^2(0,T;\mathbf{L}^2(\Omega))$, $p_d \in L^2(0,T;L^2(\Omega))$, $\chi \in L^2(\Gamma_{D,v})$, $\lambda \geq 0$, and a finite set of integers $W \subset \mathbb{Z}$. We define

$$
(5.10)\qquad J(\mathbf{u},p,w) \coloneqq \frac{1}{2}\|\mathbf{u}-\mathbf{u}_d\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2}\|p-p_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\lambda}{2}\|w\|^2_{L^2(0,T)}
$$

for $(\mathbf{u},p,w) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;V) \times L^2(0,T)$ and the reduced objective

$$
(5.11)\qquad j : L^2(0,T) \to \mathbb{R},\ \ j(w) \coloneqq J(Gw + (\tilde{\mathbf{u}},\tilde{p}), w)
$$

Next we verify the applicability of the SLIP algorithm introduced in the previous section, by checking if Assumption 4.3 and Assumption 4.1 are satisfied.

5.1 CASE 1: $\lambda > 0$

In this subsection, we prove that in both poroelastic and poroviscoelastic cases Assumption 4.3 is satisfied while the more stringent Assumption 4.1 is not satisfied.

Proposition 5.4. *Let $\xi \in L^2(0,T)$. Then it follows that the reduced objective $j : L^2(0,T) \to \mathbb{R}$ is twice continuously differentiable at ξ. Moreover, $|\nabla^2 j(w)(\xi,\varphi)| = |(G\xi, G\varphi) + \lambda(\xi,\varphi)|$ and there exists $C > 0$ such that*

$$
(5.12)\qquad |\nabla^2 j(w)(\xi,\varphi)| \leq C\|\xi\|_{L^2(0,T)}\|\varphi\|_{L^2(0,T)}
$$

for all $w \in L^2(0,T)$ and all $\xi, \varphi \in L^2(0,T)$ i.e. the Hessian is continuous on $L^2(0,T)\times L^2(0,T)$. When $\lambda > 0$, Assumption 4.1 does not hold.

Proof. Let $\tilde{y} = (\mathbf{u}_d - \tilde{\mathbf{u}}, p_d - \tilde{p})$. Based on the definition (5.11) of the reduced functional j, we have that

$$
\nabla j(q) = (G(\cdot), Gq - \tilde{y}) + \lambda(\cdot, q) \text{ and } \nabla^2 j(q)(\xi,\varphi) = (G\varphi, G\xi) + \lambda(\varphi,\xi)
$$

Furthermore, we can estimate the Hessian of j as follows:

$$
(5.13)\qquad \begin{aligned} \left|\nabla^2 j(q)(\xi,\varphi)\right| &= |(G\varphi, G\xi) + \lambda(\varphi,\xi)| \\ &\leq \|G\varphi\|_{L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))}\|G\xi\|_{L^2(0,T;\mathbf{L}^2(\Omega)\times L^2(\Omega))} + \lambda\|\varphi\|_{L^2(0,T)}\|\xi\|_{L^2(0,T)} \\ &\leq (C^2+\lambda)\|\varphi\|_{L^2(0,T)}\|\xi\|_{L^2(0,T)}, \end{aligned}
$$

which provides the desired estimate (5.12). Additionally, we note that

$$
(5.14)\qquad \lambda\|\varphi\|^2_{L^2(\Omega)} \leq \lambda(\varphi,\varphi) + (G\varphi, G\varphi) = \nabla^2 j(q)(\varphi,\varphi) \quad \text{for all } \varphi \in L^2(0,T).
$$

Consider the sequence $\phi_n(t) = n\left(\frac{t}{T}\right)^n \in L^2(0,T)$. Note that

$$
(5.15)\qquad \|\phi_n\|^2_{L^1(0,T)} = \left(\int_0^T \left|n\left(\frac{t}{T}\right)^n\right| dt\right)^2 = \frac{n^2T^2}{(n+1)^2}
$$

$$\lambda \|\phi_n\|^2_{L^2(0,T)} = \lambda \int_0^T \left| n \left(\frac{t}{T}\right)^n \right|^2 dt = \frac{\lambda n^2 T}{2n+1}. \tag{5.16}$$

Combining (5.14) with (5.15) and (5.16) we obtain

$$\frac{|\nabla^2 j(w)(\phi_n, \phi_n)|}{\|\phi_n\|^2_{L^1(0,T)}} \geq \frac{\lambda \|\phi_n\|^2_{L^2(\Omega)}}{\|\phi_n\|^2_{L^1(\Omega)}} = \frac{\lambda n^2 T}{2n+1} \frac{(n+1)^2}{n^2 T^2} = \lambda \frac{(n+1)^2}{(2n+1)T} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Hence, Assumption 4.1 does not hold. □

5.2 CASE 2: $\lambda = 0$

Now we consider the case $\lambda = 0$. For the following proofs and numerical results, we consider the problem in one spatial dimension i.e. $\Omega = (0, L)$. Following [7, 55], the one dimensional formulation of the poroviscoelastic systems is given by

$$H_v \frac{\partial^3 u}{\partial x^2 \partial t} + H_e \frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x} = 0 \qquad \forall (x,t) \in [0,L] \times [0,T] \tag{5.17}$$

$$\frac{\partial^2 u}{\partial x \partial t} - k \frac{\partial^2 p}{\partial x^2} = S \qquad \forall (x,t) \in [0,L] \times [0,T] \tag{5.18}$$

$$u(0,t) = p(0,t) = 0 \qquad \forall t \in [0,T] \tag{5.19}$$

$$-k \frac{\partial p}{\partial x}(L,t) = \psi(t) \qquad \forall t \in [0,T] \tag{5.20}$$

$$H_v \frac{\partial^2 u}{\partial x \partial t}(L,t) + H_e \frac{\partial u}{\partial x}(L,t) - p(L,t) = 0 \qquad \forall t \in [0,T] \tag{5.21}$$

$$\frac{\partial u}{\partial x}(x,0) = 0 \qquad \forall x \in [0,L] \tag{5.22}$$

where $H_e = \lambda_e + 2\mu_e$ and $H_v = \delta(\lambda_v + 2\mu_v)$. When $F = 0$, we know $p(x,t) = H_e \frac{\partial u}{\partial x} + H_v \frac{\partial^2 u}{\partial x \partial t} + h(t)$. When $g = 0$, applying (5.21) shows $h(t)$ must be equal to 0. Therefore, when $F = 0$ and $g = 0$ we have

$$p(x,t) = H_e \frac{\partial u}{\partial x} + H_v \frac{\partial^2 u}{\partial x \partial t}. \tag{5.23}$$

When $\lambda = 0$, we will see that Assumption 4.1 holds in the poroelastic case, but not the poroviscoelastic case. We will study these two cases separately.

5.2.1 *POROELASTIC CASE*

We will first consider the case when ψ is used as the control. Then we will study the case where the control $w(t)$ enters the system in the source S, i.e. $S(x,t) = \chi(x) w(t)$.

Theorem 5.5. *Let $\lambda = 0$ and $\delta = 0$. Let $k(x,t) = k$ where k is a positive constant and let G map a control ψ to the state (u,p) that satisfies (5.17)-(5.22) with all sources set to zero except $\psi(t)$. Then Assumption 4.1 holds with $w = \psi$.*

Proof. Recall from (5.13), that $\nabla^2 j(\psi)(\xi, \varphi) = (G\xi, G\varphi)$. We will first calculate $G\xi$ when $\xi \in C^1_0(0,T)$. Then we will use the Bounded Linear Extension Theorem to show Assumption 4.1 holds. Note that ξ is only a function of t because we are considering the one-dimensional case where Γ_N is only one point.

Since $\delta = 0$, $H_v = 0$. Plugging (5.23) into the PDE, we see that $p(x,t)$ needs to satisfy

$$\begin{aligned} \frac{\partial p}{\partial t} - kH_e\frac{\partial^2 p}{\partial x^2} &= 0 && \forall (x,t) \in [0,L]\times[0,T] \\ p(0,t) = 0, \quad -k\frac{\partial p}{\partial x}(L,t) &= \xi(t) && \forall t \in [0,T] \\ p(x,0) &= 0 && \forall x \in [0,L]. \end{aligned}$$

Let $\rho = p + \frac{x}{k}\xi(t)$. Then, we see

$$\begin{aligned} \frac{\partial \rho}{\partial t} - kH_e\frac{\partial^2 \rho}{\partial x^2} &= \frac{x}{k}\xi'(t) && \forall (x,t) \in [0,L]\times[0,T] \\ \rho(0,t) = 0, \quad -k\frac{\partial \rho}{\partial x}(L,t) &= 0 && \forall t \in [0,T] \\ \rho(x,0) &= 0 && \forall x \in [0,L]. \end{aligned}$$

Let $\lambda_n = \frac{(2n-1)\pi}{2L}$. Note that $(\sqrt{2/L}\sin(\lambda_n x))_{n\in\mathbb{N}}$ is a complete orthonormal basis. Let

$$\rho(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x).$$

Then we see $\rho(x,t)$ satisfies all boundary conditions, and we have

$$\sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^2 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) = \frac{x}{k}\xi'(t)$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x) = 0.$$

Multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, and setting $c_n = (\frac{x}{k}, \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)} = \frac{(-1)^n\sqrt{2L}}{k\lambda_n^2}$, we have

$$f_n'(t) + kH_e\lambda_n^2 f_n(t) = c_n\xi'(t) \quad \text{and} \quad f_n(0) = 0.$$

Therefore, $f_n(t) = e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\xi'(\xi)d\xi$. Hence, we have

$$\begin{aligned} \rho(x,t) &= \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\xi'(\xi)d\xi\sqrt{2/L}\sin(\lambda_n x), \\ p(x,t) &= \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\xi'(\xi)d\xi\sqrt{2/L}\sin(\lambda_n x) - \frac{x}{k}\xi(t). \end{aligned}$$

Using (5.23) and recalling that $H_v = 0$, we anti-differentiate p with respect to x and enforce the boundary condition $u(x,0) = 0$, to see

$$u(x,t) = \sum_{n=1}^{\infty} \frac{e^{-kH_e\lambda_n^2 t}}{H_e\lambda_n}\int_0^t c_n e^{kH_e\lambda_n^2\xi}\xi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x)) - \frac{x^2}{2kH_e}\xi(t).$$

Using the fact that $c_n = (\frac{x}{k}, \sqrt{2/L}\sin(\lambda_n x))_{L^2(\Omega)}$, $(\sqrt{2/L}\sin(\lambda_n x))_{n\in\mathbb{N}}$ is an orthonormal sequence, Lebesgue Dominated Convergence Theorem, and Parseval's equality, we have

$$
\begin{aligned}
&\|p\|^2_{L^2(0,T;L^2(0,L))}\\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2 \left[\left(e^{-kH_e\lambda_n^2 t} \int_0^t e^{kH_e\lambda_n^2 \xi} \xi'(\xi) d\xi \right)^2 - 2e^{-kH_e\lambda_n t} \int_0^t e^{kH_e\lambda_n^2\xi} \xi'(\xi) d\xi \xi(t) \right] + \frac{x^2}{k^2} \xi^2(t) dt \\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2 \left[\left(e^{-kH_e\lambda_n^2 t} \int_0^t e^{kH_e\lambda_n^2 \xi} \xi'(\xi) d\xi \right)^2 - 2e^{-kH_e\lambda_n t} \int_0^t e^{kH_e\lambda_n^2\xi} \xi'(\xi) d\xi \xi(t) + \xi^2(t) \right] dt.
\end{aligned}
$$

Integrating by parts and using $\xi(0) = 0$, we see

$$
\begin{aligned}
\|p\|^2_{L^2(0,T;L^2(0,L))} &= \int_0^T \sum_{n=1}^{\infty} c_n^2 \left[\left(e^{-kH_e\lambda_n^2 t} \left(\xi(t) e^{kH_e\lambda_n^2 t} - \int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2\xi} d\xi \right) \right)^2 \right.\\
&\left. -2e^{-kH_e\lambda_n^2 t} \left(\xi(t) e^{kH_e\lambda_n^2 t} - \int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2\xi} d\xi \right) \xi(t) + \xi^2(t) \right] dt \\
&= \int_0^T \sum_{n=1}^{\infty} c_n^2 \left[\xi^2(t) - 2\xi(t) \int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi + \left(\int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi \right)^2 - 2\xi^2(t) \right.\\
&\left. +2\xi(t) \int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi + \xi^2(t) \right] dt = \int_0^T \sum_{n=1}^{\infty} c_n^2 \left(\int_0^t \xi(\xi) kH_e\lambda_n^2 e^{kH_e\lambda_n^2(\xi-t)} d\xi \right)^2 dt \\
&\le \int_0^T \sum_{n=1}^{\infty} c_n^2 k^2 H_e^2 \lambda_n^4 \left(\int_0^T \xi(\xi) e^{-kH_e\lambda_n^2(t-\xi)} d\xi \right)^2 dt \le \int_0^T \sum_{n=1}^{\infty} H_e^2 \left(\int_0^T \xi(\xi) e^{-kH_e\lambda_n^2(t-\xi)} d\xi \right)^2 dt.
\end{aligned}
$$

Using $p = 1$, $q = 2$, and $r = 2$ in (2.1), we have

$$
\begin{aligned}
\|p\|^2_{L^2(0,T;L^2(\Omega))} &\le \sum_{n=1}^{\infty} H_e^2 \|e^{-kH_e\lambda_n^2(\cdot)}\|^2_{L^2(0,T)} \|\xi\|^2_{L^1(0,T)} \le \sum_{n=1}^{\infty} \frac{H_e^2}{2kH_e\lambda_n^2} \|\xi(\xi)\|^2_{L^1(0,T)} \\
&= \frac{H_e}{2k} \|\xi\|^2_{L^1(0,T)} \sum_{n=1}^{\infty} \frac{4L^2}{(2n-1)^2\pi^2} = \frac{H_e L^2}{4k} \|\xi\|^2_{L^1(0,T)}
\end{aligned}
$$

Additionally, integrating (5.23) with respect to x, we have

$$
\|u\|^2_{L^2(0,T;L^2(0,L))} = \int_0^T \int_0^L \left(\int_0^x p(\xi, t) d\xi \right)^2 dx dt \le \frac{H_e L^3}{4k} \|\xi\|^2_{L^1(0,T)}
$$

Therefore, for all $\xi \in C_0^1(0,T)$, we have

$$
\|G\xi\|^2_{L^2(0,T;L^2(0,L)\times L^2(0,L))} \le C\|\xi\|^2_{L^1(0,T)}. \tag{5.24}
$$

Hence, $G|_{C_0^1(0,T)}$ is bounded and linear. Using the Bounded Linear Extension theorem, we have that the extension $G : L^1(0,T) \to L^2(0,T; L^2(0,L) \times L^2(0,L))$ is a bounded linear functional. Hence, (5.24) holds for all $\xi \in L^1(0,T)$. Thus, $|\nabla^2 j(\psi)(\xi, \varphi)| = (G\xi, G\varphi) \le \|G\xi\|_{L^2(0,T;L^2(0,L)\times L^2(0,L))} \|G\varphi\|_{L^2(0,T;L^2(0,L)\times L^2(0,L))} \le C^2 \|\xi\|_{L^1(0,T)} \|\varphi\|_{L^1(0,T)}$. □

Now we consider the poroelastic case with $\lambda = 0$ where the control is used in the source S. Let $\chi(x)$ be set and consider the case where G maps $s \in L^2(0,T)$ to the unique solution (u, p) to (5.1)-(5.5) with all sources and the initial condition set to zero except $S := \chi(x)s(t)$. In this case, the process of finding

$p(x,t)$ is similar to the process of finding $\rho(x,t)$ in the proof of Theorem 5.5. Hence, $Gs = (u,p)$ where

$$p(x,t) = \sum_{n=1}^{\infty} e^{-kH_e\lambda_n^2 t} \int_0^t d_n e^{kH_e\lambda_n^2\xi} s(\xi) d\xi \sqrt{2/L}\sin(\lambda_n x),$$

and applying (5.23)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{e^{-kH_e\lambda_n^2 t}}{H_e\lambda_n} \int_0^t d_n e^{kH_e\lambda_n^2\xi} s(\xi) d\xi \sqrt{2/L}\sin(\lambda_n x)$$

where $d_n = (\chi(x), \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)}$. The proof for showing this solution also satisfies Assumption 4.1 follows similarly to the proof of Theorem 5.5.

5.2.2 Poroviscoelastic case

We show that Assumption 4.1 is not satisfied in the poroviscoelastic case, for both choices of controls ψ and S.

Theorem 5.6. *Let $\lambda = 0$ and $\delta > 0$. Let $k(x,t) = k$ where k is a positive constant and let G map a control ψ to the state (u,p) that satisfies (5.17)-(5.22) with all sources set to zero except $\psi(t)$. Then Assumption 4.1 does not hold.*

Proof. We will proceed with this proof by first finding $G\psi = (u,p)$ when $\psi(t) \in C_0^1(0,T)$. We want to show that there exists $(\varphi_m)_{m\in\mathbb{N}}$ such that

$$\frac{(G\varphi_m, G\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{\|p_m\|^2_{L^2(0,T)} + \|u_m\|^2_{L^2(0,T)}}{\|\varphi_m\|^2_{L^1(0,T)}} \to \infty \text{ as } m \to \infty.$$

Plugging (5.23) into (5.17)-(5.22), we see that $u(x,t)$ needs to satisfy

$$\begin{aligned}
\frac{\partial^2 u}{\partial x\partial t} - kH_v\frac{\partial^4 u}{\partial x^3\partial t} - kH_e\frac{\partial^3 u}{\partial x^3} &= 0 && \forall (x,t) \in [0,L)\times[0,T]\\
u(0,t) &= 0 && \forall t \in [0,T]\\
H_v\frac{\partial^2 u}{\partial x\partial t}(0,t) + H_e\frac{\partial u}{\partial x}(0,t) &= 0 && \forall t \in [0,T]\\
-kH_v\frac{\partial^3 u}{\partial x^2\partial t}(L,t) - kH_e\frac{\partial^2 u}{\partial x^2}(L,t) &= \psi(t) && \forall t \in [0,T]\\
\frac{\partial u}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $y(x,t) = u(x,t) + \frac{x^2}{2L}\Psi(t)$ where

$$\Psi(t) = \frac{1}{kH_v} e^{-H_e/H_v t} \int_0^t \psi(\xi) e^{H_e/H_v\xi} d\xi.$$

Then $y(x,t)$ satisfies

$$\begin{aligned}
\frac{\partial^2 y}{\partial x \partial t} - kH_v \frac{\partial^4 y}{\partial x^3 \partial t} - kH_e \frac{\partial^3 y}{\partial x^3} &= \frac{x}{L}\Psi'(t) && \forall (x,t) \in [0,L] \times [0,T] \\
y(0,t) &= 0 && \forall t \in [0,T] \\
H_v \frac{\partial^2 y}{\partial x \partial t}(0,t) + H_e \frac{\partial y}{\partial x}(0,t) &= 0 && \forall t \in [0,T] \\
-kH_v \frac{\partial^3 y}{\partial x^2 \partial t}(L,t) - kH_e \frac{\partial^2 y}{\partial x^2}(L,t) &= 0 && \forall t \in [0,T] \\
\frac{\partial y}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $y(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}(1-\cos(\lambda_n x))$ where $\lambda_n = \frac{(2n-1)\pi}{2L}$. Notice that $y(0,t) = 0$, $\frac{\partial y}{\partial x}(0,t) = 0$, and $\frac{\partial^2 y}{\partial x^2}(L,t) = 0$. By plugging $y(x,t)$ into the first line of the PDE, we see that

$$\sum_{n=1}^{\infty} f_n'(t)\lambda_n\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_v \sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) = \frac{x}{L}\Psi'(t)$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x) = 0.$$

Note that $\left(\sqrt{2/L}\sin(\lambda_n x)\right)_{n\in\mathbb{N}}$ is a complete orthonormal basis. Hence, multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, setting $c_n = (\frac{x}{L}, \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)} = 4\sqrt{2L}(-1)^{n+1}/((2n-1)^2\pi^2)$ we have

$$(\lambda_n + \lambda_n^3 kH_v) f_n'(t) + kH_e\lambda_n^3 f_n(t) = c_n\Psi'(t) \quad \text{and} \quad f_n(0) = 0.$$

Therefore, $f_n(t) = \frac{1}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{(kH_e\lambda_n^2\gamma_n\xi)} c_n\Psi'(\xi)d\xi$ where $\gamma_n = \frac{1}{1+\lambda_n^2 kH_v}$. Hence,

$$\begin{aligned}
y(x,t) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi} c_n\Psi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x)) \\
u(x,t) &= \sum_{n=1}^{\infty} \frac{\gamma_n}{\lambda_n} e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi} c_n\Psi'(\xi)d\xi\sqrt{2/L}(1-\cos(\lambda_n x)) - \frac{x^2}{2L}\Psi(t)
\end{aligned}$$

and from (5.23),

$$\begin{aligned}
p(x,t) &= -H_e\frac{x}{L}\Psi(t) - H_v\frac{x}{L}\Psi'(t) + \sum_{n=1}^{\infty} c_n\lambda_n\sqrt{2/L}\sin(\lambda_n x) \\
&\quad \left(-kH_eH_v\lambda_n\gamma_n^2 e^{-kH_e\gamma_n\lambda_n^2 t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi + \frac{H_v\gamma_n}{\lambda_n}\Psi'(t) + \frac{H_e\gamma_n}{\lambda_n} e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi\right) \\
&= -\frac{x}{L}(H_e\Psi(t) + H_v\Psi'(t)) + \sum_{n=1}^{\infty} c_n\sqrt{2/L}\lambda_n\sin(\lambda_n x)\left(\frac{H_e\gamma_n^2}{\lambda_n} e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi + \frac{H_v\gamma_n}{\lambda_n}\Psi'(t)\right).
\end{aligned}$$

We notice

$$\Psi'(t) = \frac{1}{kH_v}\left(\frac{-H_e}{H_v} e^{-H_e/H_v t}\int_0^t \psi(\xi)e^{H_e/H_v\xi}d\xi + \psi(t)\right). \tag{5.25}$$

Hence,

$$p(x,t) = -\frac{x}{Lk}\psi(t) + \sum_{n=1}^{\infty} c_n\sqrt{2/L}\sin(\lambda_n x)\left(H_e\gamma_n^2 e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi + H_v\gamma_n\Psi'(t)\right).$$

Therefore,

$$\begin{aligned}\|p\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \sum_{n=1}^{\infty} c_n^2\left(H_e\gamma_n^2 e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi + H_v\gamma_n\Psi'(t)\right)^2\\ &- 2\frac{c_n^2}{k}\psi(t)\left(H_e\gamma_n^2 e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi + H_v\gamma_n\Psi'(t)\right) + \frac{c_n^2}{k^2}\psi^2(t)dt.\end{aligned}$$

Recalling (5.25), we see that when $\psi = \varphi_m = e^{mt}$, we have

$$\begin{aligned}\Psi'(t) &= \frac{1}{kH_v}\left(-\frac{H_e}{H_v}e^{-H_e/H_v t}\int_0^t e^{m\xi}e^{H_e/H_v\xi}d\xi + e^{mt}\right)\\ &= \frac{1}{kH_v}\left(-\frac{H_e}{H_v}\left(\frac{e^{mt}}{m+H_e/H_v} - \frac{e^{-H_e/H_v t}}{m+H_e/H_v}\right) + e^{mt}\right)\\ &= \frac{1}{kH_v}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H_e} - \frac{H_e e^{mt}}{mH_v+H_e} + e^{mt}\right)\\ &= \frac{1}{kH_v}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H_e} + \frac{H_v m e^{mt}}{H_v m+H_e}\right) \geq 0\end{aligned} \tag{5.26}$$

Hence,

$$\begin{aligned}&e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi\\ &= \frac{1}{kH_v(mH_v+H_e)}\left(\frac{H_e}{kH_e\lambda_n^2\gamma_n - H_e/H_v}\left(e^{-H_e/H_v t} - e^{-kH_e\lambda_n^2\gamma_n t}\right) + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t}\right)\right)\\ &= \frac{1}{kH_v(mH_v+H_e)}\left(\frac{H_v}{k\lambda_n^2\gamma_n H_v - 1}\left(e^{-H_e/H_v t} - e^{-kH_e\lambda_n^2\gamma_n t}\right) + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t}\right)\right)\end{aligned}$$

Recall $\gamma_n = \frac{1}{1+\lambda_n^2 kH_v}$, so

$$\begin{aligned}k\lambda_n^2\gamma_n H_v - 1 &= \frac{k\lambda_n^2 H_v}{1+\lambda_n^2 kH_v} - \frac{1+\lambda_n^2 kH_v}{1+\lambda_n^2 kH_v}\\ &= \frac{-1}{1+\lambda_n^2 kH_v} = -\gamma_n\end{aligned}$$

and

$$-kH_e\lambda_n^2\gamma_n = -\frac{kH_e\lambda_n}{1+\lambda_n^2 kH_v} > -\frac{kH_e\lambda_n^2}{\lambda_n^2 kH_v} = -\frac{H_e}{H_v},$$

which implies $1 \geq e^{-kH_e\lambda_n^2\gamma_n t} - e^{-H_e/H_v t} \geq 0$ for $t \in [0,T]$. Hence,

$$e^{-kH_e\lambda_n^2\gamma_n t}\int_0^t e^{kH_e\lambda_n^2\gamma_n\xi}\Psi'(\xi)d\xi \geq \frac{1}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_n} + \frac{H_v m}{kH_e\lambda_n^2\gamma_n + m}\left(e^{mt} - e^{-kH_e\lambda_n^2\gamma_n t}\right)\right) \tag{5.27}$$

Let $l > 0$ satisfy

$$\lambda_l^2 = \frac{(2l-1)^2\pi^2}{4L^2} > \frac{8}{kH_v}. \text{ Then } \gamma_l = \frac{1}{1+\lambda_l^2 kH_v} < \frac{1}{\lambda_l^2 kH_v} < \frac{k}{8}. \tag{5.28}$$

Notice that

$$\begin{aligned}\|p_m\|^2_{L^2(0,T;L^2(\Omega))} \geq \int_0^T & c_l^2 \left(H_e\gamma_l^2 e^{-kH_e\lambda_l^2\gamma_l t}\int_0^t e^{kH_e\lambda_l^2\gamma_l\xi}\Psi'(\xi)d\xi + H_v\gamma_l\Psi'(t)\right)^2 \\ & - 2\frac{c_l^2}{k}\psi(t)\left(H_e\gamma_l^2 e^{-kH_e\lambda_l^2\gamma_l t}\int_0^t e^{kH_e\lambda_l^2\gamma_l\xi}\Psi'(\xi)d\xi + H_v\gamma_l\Psi'(t)\right) + \frac{c_l^2}{k^2}\psi^2(t)dt\end{aligned}$$

Dropping the first term since it is positive and applying (5.27) and (5.26) we see

$$\begin{aligned}\|p_m\|^2_{L^2(0,T;L^2(\Omega))} \geq \int_0^T & -2\frac{c_l^2}{k}e^{mt}\frac{H_e\gamma_l^2}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_l} + \frac{H_v m}{kH_e\lambda_l^2\gamma_l + m}\left(e^{mt} - e^{-kH_e\lambda_l^2\gamma_l t}\right)\right) \\ & - 2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H_e} + \frac{H_v m e^{mt}}{H_v m + H_e}\right) + \frac{c_l^2}{k^2}e^{2mt}dt\end{aligned}$$

Recalling that $-(e^{mt} - e^{-kH_e\lambda_l^2\gamma_l t}) \geq -e^{mt}$, $-\left(\frac{H_e e^{-H_e/H_v t}}{mH_v+H+e} + \frac{H_v m e^{mt}}{H_v m+H_e}\right) \geq -(1+e^{mt})$, and $mH_v + H_e \geq H_e$ we see

$$\begin{aligned}\|p_m\|^2_{L^2(0,T;L^2(\Omega))} &\geq \int_0^T -2\frac{c_l^2}{k}e^{mt}\frac{H_e\gamma_l^2}{kH_v(mH_v+H_e)}\left(\frac{H_v}{\gamma_l} + \frac{H_v m}{m}e^{mt}\right) - 2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}(1+e^{mt}) + \frac{c_l^2}{k^2}e^{2mt}dt \\ &\geq \int_0^T -2\frac{c_l^2}{k}e^{mt}\left(\frac{H_e\gamma_l^2}{kH_vH_e}\frac{H_v}{\gamma_l} + \frac{H_e\gamma_l^2 H_v}{kH_vH_e}e^{mt}\right) - 2\frac{c_l^2}{k}e^{mt}\frac{\gamma_l}{k}(1+e^{mt}) + \frac{c_l^2}{k^2}e^{2mt}dt \\ &\geq \int_0^T -2\frac{c_l^2}{k^2}e^{2mt}\left(\gamma_l + \gamma_l^2\right) - 2\frac{c_l^2}{k^2}e^{2mt}2\gamma_l + \frac{c_l^2}{k^2}e^{2mt}dt\end{aligned}$$

Hence, applying (5.28), we see

$$\|p_m\|^2_{L^2(0,T;L^2(\Omega))} \geq \int_0^T -2\frac{c_l^2}{k^2}e^{2mt}\left(\frac{1}{8}+\frac{1}{64}+\frac{1}{4}\right) + \frac{c_l^2}{k^2}e^{2mt}dt \geq \int_0^T \frac{7c_l^2}{32^2}e^{2mt}dt = \frac{7c_l^2}{64k^2m}\left(e^{2mT}-1\right). \tag{5.29}$$

Also, note $\|\varphi_m\|^2_{L^1(0,T)} = \frac{(e^{mt}-1)^2}{m^2}$. Therefore,

$$\begin{aligned}\frac{\nabla^2 j(w)(\varphi_m,\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{(G\varphi_m, G\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} &= \frac{\|p_m\|^2_{L^2(0,T;L^2(0,L))} + \|u_m\|^2_{L^2(0,T;L^2(0,L))}}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{\|p_m\|^2_{L^2(0,T;L^2(0,L))}}{\|\varphi_m\|^2_{L^1(0,T)}} \\ &\geq \frac{\frac{7c_l}{64mk^2}(e^{2mT}-1)m^2}{(e^{mT}-1)^2} \to \infty \text{ as } m \to \infty.\end{aligned}$$

Therefore, Assumption 4.1 is not satisfied. □

We now consider the poroviscoelastic case where the control is used as the time component of the source S, and show Assumption 4.1 is not satisfied.

Theorem 5.7. *Let $\lambda = 0$ and $\delta > 0$. Let $k(x,t) = k$ and $\chi(x) \in L^2(0,L)$. Define $G : L^2(0,T) \to L^2(0,T;L^2(0,L)\times L^2(0,L))$ to be the map that maps $s(t) \in L^2(0,T)$ to the (u,p) that satisfies (5.17)-(5.22) with all sources set to 0 except $S(x,t) = s(t)\chi(x)$. Then Assumption 4.1 does not hold.*

Proof. We will proceed with this proof by first computing $Gs = (u, p)$ and then showing a lower estimate on $\|p\|^2_{L^2(0,T;L^2(\Omega))}$. Plugging (5.23) into the PDE we see $u(x,t)$ needs to satisfy

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial t} - kH_v \frac{\partial^4 u}{\partial x^3 \partial t} - kH_e \frac{\partial^3 u}{\partial x^3} &= S && \forall (x,t) \in [0,L) \times [0,T] \\
u(0,t) &= 0 && \forall t \in [0,T] \\
H_v \frac{\partial^2 u}{\partial x \partial t}(0,t) + H_e \frac{\partial u}{\partial x}(0,t) &= 0 && \forall t \in [0,T] \\
-kH_v \frac{\partial^3 u}{\partial x^2 \partial t}(L,t) - kH_e \frac{\partial^2 u}{\partial x^2}(L,t) &= 0 && \forall t \in [0,T] \\
\frac{\partial u}{\partial x}(x,0) &= 0 && \forall x \in [0,L]
\end{aligned}$$

Let $u(x,t) = \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}(1 - \cos(\lambda_n x))$, where $\lambda_n = \frac{(2n-1)\pi}{2L}$. Notice that $u(0,t) = 0$, $\frac{\partial u}{\partial x}(0,t) = 0$, and $\frac{\partial^2 u}{\partial x^2}(L,t) = 0$, so the boundary conditions are satisfied. We have

$$\sum_{n=1}^{\infty} f_n'(t)\lambda_n \sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_v \sum_{n=1}^{\infty} f_n'(t)\sqrt{2/L}\sin(\lambda_n x) + \lambda_n^3 kH_e \sum_{n=1}^{\infty} f_n(t)\sqrt{2/L}\sin(\lambda_n x) = s(t)\chi(x)$$

and

$$\sum_{n=1}^{\infty} f_n(0)\sqrt{2/L}\sin(\lambda_n x) = 0.$$

Note $\left(\sqrt{2/L}\sin(\lambda_n x)\right)_{n\in\mathbb{N}}$ is a complete orthonormal basis. Hence, multiplying both sides of these equations by $\sqrt{2/L}\sin(\lambda_n x)$, integrating these equations from 0 to L, and setting $c_n = (\chi(x), \sqrt{2/L}\sin(\lambda_n x))_{L^2(0,L)}$, we have for all $n \in \mathbb{N}$

$$(\lambda_n + \lambda_n^3 kH_v) f_n'(t) + kH_e \lambda_n^3 f_n(t) = c_n s(t) \quad \text{and} \quad f_n(0) = 0.$$

Therefore, $f_n(t) = \frac{1}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t c_n e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi \sqrt{2/L}(1 - \cos(\lambda_n x))$$

and (5.23) gives

$$
\begin{aligned}
p(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x) \Bigg(& \frac{-kH_eH_v\lambda_n}{(1+\lambda_n^2 kH_v)^2} e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi \\
& + \frac{H_v}{\lambda_n + \lambda_n^3 kH_v} s(t) + \frac{H_e}{\lambda_n + \lambda_n^3 kH_v} e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi \Bigg) \\
= \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x) \Bigg(& \left(\frac{-kH_eH_v\lambda_n^2}{\lambda_n(1+\lambda_n^2 kH_v)^2} + \frac{H_e(1+\lambda_n^2 kH_v)}{\lambda_n(1+\lambda_n^2 kH_v)^2} \right) e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \\
& \int_0^t e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi + \frac{H_v}{\lambda_n + \lambda_n^3 kH_v} s(t) \Bigg) \\
= \sum_{n=1}^{\infty} c_n \sqrt{2/L}\sin(\lambda_n x) \Bigg(& \frac{H_e}{\lambda_n(1+\lambda_n^2 kH_v)^2} e^{-kH_e\lambda_n^2 t/(1+\lambda_n^2 kH_v)} \int_0^t e^{kH_e\lambda_n^2 \xi/(1+\lambda_n^2 kH_v)} s(\xi) d\xi \\
& + \frac{H_v}{\lambda_n + \lambda_n^3 kH_v} s(t) \Bigg).
\end{aligned}
$$

Therefore, when $s(t)$ is strictly non-negative,

$$
\|p\|^2_{L^2(0,T;L^2(\Omega))} \geq \sum_{n=1}^{\infty} \frac{c_n^2 H_v^2}{(\lambda_n + \lambda_n^3 kH_v)^2} \|s\|^2_{L^2(0,T)} \geq C\|s\|^2_{L^2(0,T)}
$$

for some $C > 0$. Hence, using $\varphi_m = m\left(\frac{t}{T}\right)^m$ (as was done in the proof of Proposition 5.4), we have

$$
\frac{\nabla^2 j(w)(\varphi_m, \varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} = \frac{(G\varphi_m, G\varphi_m)}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{\|p_m\|_{L^2(0,T;L^2(\Omega))}}{\|\varphi_m\|^2_{L^1(0,T)}} \geq \frac{C\|\varphi_m\|_{L^2(0,T)}}{\|\varphi_m\|^2_{L^1(0,T)}} \to \infty \quad \text{as } m \to \infty.
$$

Therefore, Assumption 4.1 is not satisfied. □

6 COMPUTATIONAL EXPERIMENTS

For our computational experiments, we use an instance of the one-dimensional porous medium equations described in Section 5. We intend to analyze the effect of the mollification regularization on the resulting instationarity and the objective values in practice. Specifically, we consider the control input ψ, where Assumption 4.1 is satisfied for the poroelastic case and violated for the poroviscoelastic case. In Section 6.1, the specific model as well as the discretization of the model and the trust-region subproblems are described as well as the details for the computational experiements and the homotopy. We present and describe our results in Section 6.2.

6.1 NUMERICAL EXPERIMENTS

We consider a discretized instance of (P) that is governed by the PDE introduced in Section 5 with a one-dimensional spatial domain $\Omega = (0, 2)$ and a one-dimensional time domain $(0,T) = (0, 0.5)$. We use the same discretization as in [8]. For the time horizon we use N=512 uniform intervals and an implicit Euler scheme. For the space discretization, we use a dual hybridized finite element method with 512 uniform intervals on each of which the control is constant with a value in W. We choose the set of possible control realizations as $W = \{-7, -5, -3, -1, 0, 2\}$ in our computations.

Regarding the parameters of the PDE we use $k = 1$, $\lambda_e = 1$, $\mu_e = 1$, $\mu_v = 0.25$, $\lambda_v = 0.0774$. We execute the same experiments for both the two choices $\delta = 0$ (poroelastic case) and $\delta = 1$ (poroviscoelastic case).

Regarding the setup of the control problem, we choose the structure given in (5.9) where the PDE input choices are $\psi = w$ ($F = 0$, $S = 0$, $g = 0$) and $S = \chi w$ for a fixed function χ ($F = 0$, $g = 0$, $\psi = 0$), where w denotes the control function. We choose the penalty parameter value $\alpha = 5 \cdot 10^{-5}$ to scale the TV-term in the objective. We run all experiments for the choices $\lambda = 10^{-4}$, $\lambda = 10^{-2}$, and $\lambda = 0$. We tabulate for which of the settings Assumption 4.1 is violated or satisfied in Table 1 according to the results obtained in Section 5. We note that Assumption 4.1 is not satisfied for both $\delta = 0$ and $\delta = 1$ for

Table 1: Satisfaction (T) and violation (F) of Assumption 4.1 for the different experiments.

	Input is S			Input is ψ		
$\lambda =$	0	10^{-4}	10^{-2}	0	10^{-4}	10^{-2}
$\delta = 0$	T	F	F	T	F	F
$\delta = 1$	F	F	F	F	F	F

$\lambda = 10^{-2}$ and $\lambda = 10^{-4}$ by virtue of Proposition 5.4 regardless of the fact which of the control inputs is chosen. Additionally, since (5.13) shows Assumption 4.3 is satisfied in all cases, we note that Proposition 4.5 implies Assumption 4.1 is satisfied for j_ϵ in all the settings.

For the tracking terms, we choose $u_d(x,t) = 0.5 + (1-t)^2 \cos(50t)(-1.975x + 4)$ and $p_d(x,t) = 0.5 + \cos(50t)^2$ for all $(x,t) \in \Omega \times (0,T)$. The tracking-type term and the squared L^2-norm of the control, are discretized using the trapezoidal rule for the same intervals. The derivative of the of the first (reduced) term of the objective is required to evaluate the linear part of the objective of the subproblem (TR). In order to compute the latter, we use a first discretize, then optimize-based [33] adjoint calculus.

In our executions of Algorithm 1 on a computer, we select six feasible initial controls w^0, specifically $w^0 \equiv \omega$ for all $\omega \in W$. Then, we replace the infinite-dimensional trust-region subproblems with the discretizations that are described above. We note that we have the (implicit) termination criterion in Algorithm 1 that the trust-region radius contracts to a value below T/N because we operate with limited precision and a fixed discretization. In this case, the linear integer program that arises after discretizing (TR) has only one feasible point, namely the function $d = 0$ with objective value 0. Thus we always run Algorithm 1 until this situation occurs. The reset trust-region radius is $\Delta^0 = 0.25T$. The acceptance value for the ratio of actual over predicted reduction is $\sigma = 10^{-3}$.

Algorithm 1 is implemented in MATLAB. C++ is used for the subproblem solver implementation, which follows [53]. All computations were executed on a workstation with an AMD Epic 7742 CPU and 96 GB RAM.

For each of the 6 initializations w^0, we record the final control w^f, the final objective value $j(w^f) + \alpha\,\mathrm{TV}(w^f)$, and the instationarity $C(w^f)$ on termination for all of these executions. Here, $C(w^f) := \|(\nabla j(w^f)(t_i))_{i=1}^{\#s}\|$, where $t_1, \ldots, t_{\#s}$ for some $\#s \in \mathbb{N}$ denote the switching times of w^f, that is the values $\hat{t} \in (0,T)$ such that $\lim_{t\downarrow\hat{t}} w^f(t) \neq \lim_{t\uparrow\hat{t}} w^f(t)$. Note again that the stationarity condition from Definition 3.1 becomes $\nabla j(w^f)(t_i) = 0$ for all such switching times $t_1, \ldots, t_{\#s}$ if $\nabla j(w^f)$ is a continuous function, which is ensured by the regularity of the solution of the adjoint equation. Consequently, the instationarity is the norm of the vector of the $\#s$ individual violations of this instationarity condition.

Then we regularize j by composing it with the application of a standard mollifier to the control input following our recipe in Section 4, that is we replace j by $j \circ K_\varepsilon$ in (P) and ∇j by $K_\varepsilon^*(\nabla j) \circ K_\varepsilon$ in (TR). For each of the six initial controls w^0 we execute a homotopy of ε and Algorithm 1 on the regularized problems with the following regularization parameter values

$$\varepsilon \in \{1.6 \times 10^{-2}, 8 \times 10^{-3}, 4 \times 10^{-3}, 2 \times 10^{-3}, 1 \times 10^{-3}, 5 \times 10^{-4}, 2.5 \times 10^{-4}, 0\}.$$

We initialize Algorithm 1 with w^0 for the largest regularization parameter value $\varepsilon = 1.6 \times 10^{-2}$ and initialize the execution of Algorithm 1 for the subsequent value of ε with the final control function iterate of the previous parameter value for ε. Again, we record the final controls, objective values, and instationarities on termination.

For $\lambda > 0$, L-stationarity is not known to be a necessary optimality condition. Moreover, the fixed discretization also implies that we cannot expect that the final iterate is perfectly L-stationary even if Assumption 4.1 is satisfied. In order to provide a full picture, we have chosen to still measure remaining instationarity for the final control iterates and report the final instationarities for the unregularized optimization and the homotopy but kindly ask the reader to take these values with caution. We assess the remaining instationarity of the final control iterate w^f by evaluating $C(w^f)$.

6.2 Results

The results achieved with the unregularized optimization differ significantly from those obtained with the homotopy. We first note that for the same initialization, the run at the end of the homotopy with $\varepsilon = 0$ and a plain run of Algorithm 1 with $\varepsilon = 0$ with the initial guess from the beginning of the homotopy produce different final iterates (approximately L-stationary points). Within the homotopy, the initial guess for an execution of Algorithm 1 is the final iterate of the previous execution of Algorithm 1 with a larger value of ε. Consequently, since the initial guesses of the two runs with $\varepsilon = 0$ are different and the problem is nonconvex, they can lead to different sequences of iterates that converge to different L-stationary points.

Input choice ψ. We report the details of the numerical results when ψ is used as the input choice in Table 2 for the poroelastic case $\delta = 0$ and in Table 3 for the poroviscoelastic case. Detailed iteration numbers over the different values of the homotopy are given in Table 4 for $\delta = 0$ and Table 5 for $\delta = 1$.

In order to also give a qualitative expression of the produced controls, we visualize them in Figures 1 and 2 for $\delta = 0$, $\delta = 1$ and $\lambda = 10^{-2}$.

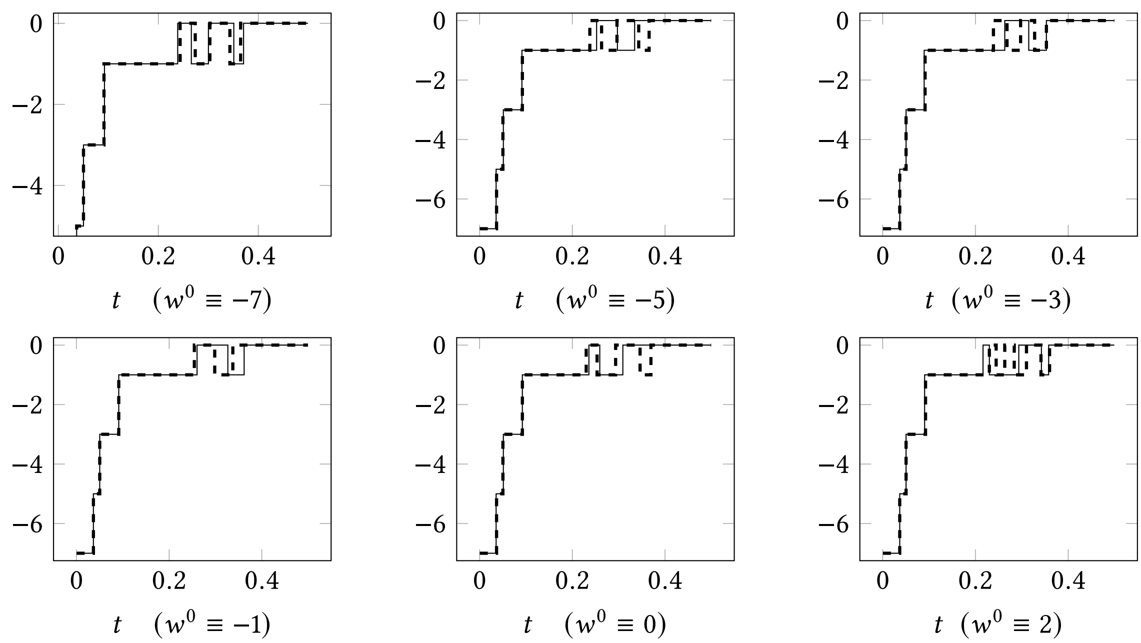

Figure 1: Control function ψ, poroelastic case $\delta = 0$, $\lambda = 10^{-2}$: final control functions produced for the unregularized optimization (solid) and for the homotopy (dashed).

Table 2: Control function ψ, poroelastic case $\delta = 0$: Objective, instationarity, and (cumulative) outer iterations of (P) for six initial controls achieved by executing Algorithm 1 on the discretized problem as well as by executing a homotopy, abbreviated (H), of executions of Algorithm 1 on regularized problems with regularization parameter driven to zero for the same initial controls.

	Ini.	Final Obj.	Final Obj. (H)	Final Inst.	Final Inst. (H)	Iter.	Cum. Iter. (H)
$\lambda = 0$	1	1.3474	1.3474	3.4×10^{-6}	2.2×10^{-6}	35	43
$\lambda = 10^{-4}$	1	1.3476	1.3476	3.0×10^{-6}	2.8×10^{-6}	35	42
$\lambda = 10^{-2}$	1	1.3616	1.3616	3.5×10^{-6}	2.2×10^{-6}	31	47
$\lambda = 0$	2	1.3474	1.3474	2.6×10^{-6}	2.1×10^{-6}	64	81
$\lambda = 10^{-4}$	2	1.3476	1.3476	3.4×10^{-6}	2.9×10^{-6}	47	79
$\lambda = 10^{-2}$	2	1.3616	1.3616	7.2×10^{-6}	3.8×10^{-6}	33	56
$\lambda = 0$	3	1.3474	1.3474	3.8×10^{-6}	2.3×10^{-6}	33	78
$\lambda = 10^{-4}$	3	1.3476	1.3476	2.1×10^{-6}	2.1×10^{-6}	38	84
$\lambda = 10^{-2}$	3	1.3616	1.3616	9.0×10^{-6}	3.6×10^{-6}	36	67
$\lambda = 0$	4	1.3475	1.3474	3.0×10^{-6}	1.4×10^{-6}	22	48
$\lambda = 10^{-4}$	4	1.3477	1.3476	3.4×10^{-6}	2.9×10^{-6}	22	43
$\lambda = 10^{-2}$	4	1.3616	1.3615	2.7×10^{-6}	3.4×10^{-6}	26	35
$\lambda = 0$	5	1.3474	1.3474	4.3×10^{-6}	2.5×10^{-6}	21	56
$\lambda = 10^{-4}$	5	1.3476	1.3476	2.3×10^{-6}	3.0×10^{-6}	26	71
$\lambda = 10^{-2}$	5	1.3616	1.3616	4.5×10^{-6}	2.1×10^{-6}	27	55
$\lambda = 0$	6	1.3474	1.3474	1.9×10^{-6}	1.3×10^{-6}	50	90
$\lambda = 10^{-4}$	6	1.3476	1.3476	2.5×10^{-6}	2.2×10^{-6}	49	69
$\lambda = 10^{-2}$	6	1.3616	1.3617	8.3×10^{-6}	5.4×10^{-6}	39	70

The unregularized optimization terminates after taking between 21 and 64 iterations for $\delta = 0$ and taking between 24 and 67 iterations for $\delta = 1$. The homotopy takes a cumulative number of iterations between 35 and 90 iterations for $\delta = 0$ and takes a much higher cumulative number between 194 and 349 iterations for $\delta = 1$.

The objective values with the unregularized optimization problem are very similar to the objective values produced by the homotopy for $\delta = 0$ with relative differences generally below 10^{-4}. This is different for $\delta = 1$, where the homotopy generally achieves lower objective values with relative differences generally around 10^{-3}.

For $\delta = 0$ the remaining instationarities are generally similar and of the same order of magnitude for the unregularized optimization and the homotopy. For $\delta = 1$, the final instationarities obtained with the unregularized optimization are generally (but not in all cases) between one and two orders of magnitude higher.

Input choice S. We report the details in Table 6 for the poroelastic case $\delta = 0$ and in Table 7 for the poroviscoelastic case. Detailed iteration numbers over the different values of the homotopy are given in Table 8 for $\delta = 0$ and Table 9 for $\delta = 1$.

The unregularized optimization terminates after taking between 12 and 29 iterations for $\delta = 0$ and taking between 8 and 21 iterations for $\delta = 1$. The homotopy takes a cumulative number of iterations between 23 and 40 iterations for $\delta = 0$ and takes a much higher cumulative number between 42 and 58 iterations for $\delta = 1$.

The objective values with the unregularized optimization problem have relative differences generally

Table 3: Control function ψ, poroviscoelastic case $\delta = 1$: Objective, instationarity, and (cumulative) outer iterations of (P) for six initial controls achieved by executing Algorithm 1 on the discretized problem as well as by executing a homotopy, abbreviated (H), of executions of Algorithm 1 on regularized problems with regularization parameter driven to zero for the same initial controls.

	Ini.	Final Obj.	Final Obj. (H)	Final Inst.	Final Inst. (H)	Iter.	Cum. Iter. (H)
$\lambda = 0$	1	1.3647	1.3625	1.7×10^{-4}	4.0×10^{-5}	48	265
$\lambda = 10^{-4}$	1	1.3647	1.3625	1.7×10^{-4}	1.1×10^{-6}	40	232
$\lambda = 10^{-2}$	1	1.3732	1.3697	2.4×10^{-4}	5.8×10^{-5}	35	240
$\lambda = 0$	2	1.3646	1.3624	1.1×10^{-4}	1.3×10^{-6}	62	289
$\lambda = 10^{-4}$	2	1.3644	1.3625	1.1×10^{-4}	1.1×10^{-6}	57	278
$\lambda = 10^{-2}$	2	1.3715	1.3696	9.2×10^{-5}	1.8×10^{-6}	67	349
$\lambda = 0$	3	1.3657	1.3624	1.1×10^{-4}	1.3×10^{-6}	26	268
$\lambda = 10^{-4}$	3	1.3656	1.3625	1.1×10^{-4}	1.1×10^{-6}	31	262
$\lambda = 10^{-2}$	3	1.3730	1.3696	1.1×10^{-4}	1.8×10^{-6}	26	241
$\lambda = 0$	4	1.3624	1.3624	1.3×10^{-6}	1.3×10^{-6}	24	201
$\lambda = 10^{-4}$	4	1.3625	1.3625	1.1×10^{-6}	1.1×10^{-6}	22	207
$\lambda = 10^{-2}$	4	1.3696	1.3696	1.8×10^{-6}	1.8×10^{-6}	22	194
$\lambda = 0$	5	1.3652	1.3624	7.5×10^{-5}	1.3×10^{-6}	34	240
$\lambda = 10^{-4}$	5	1.3652	1.3625	7.5×10^{-5}	1.1×10^{-6}	34	264
$\lambda = 10^{-2}$	5	1.3716	1.3696	6.7×10^{-5}	1.8×10^{-6}	41	283
$\lambda = 0$	6	1.3634	1.3624	1.1×10^{-4}	1.3×10^{-6}	43	303
$\lambda = 10^{-4}$	6	1.3635	1.3625	1.1×10^{-4}	1.1×10^{-6}	43	312
$\lambda = 10^{-2}$	6	1.3700	1.3696	6.3×10^{-5}	1.8×10^{-6}	64	349

around 10^{-2} compared to the objective values produced by the homotopy for $\delta = 0$. This is different for $\delta = 1$, where the homotopy generally achieves lower objective values with relative differences generally (but not in all cases) higher than 10^{-1}.

For $\delta = 0$ the remaining instationarities are generally similar and of the same order of magnitude for the unregularized optimization and the homotopy. For $\delta = 1$, the final instationarities obtained with the unregularized optimization are generally (but not in all cases) between one and two orders of magnitude higher.

7 CONCLUSION

We investigated the regularity condition Assumption 4.1 that is required for (P) for the convergence analysis of Algorithm 1 in [40] and proved a Γ-convergence result on a mollification of the control input as well as strict convergence for the iterates of a corresponding homotopy trust-region algorithm.

We assess the proposed regularization for control problems governed by poroelastic and poroviscoelastic equations modeling fluid flows through porous media. We considered the associated 1D (in space) models with two possible control inputs (one acting in the interior and one acting on the boundary). We showed that the regularity conditions are violated when the viscosity parameter δ is taken strictly greater than zero or a Tikhonov term (for example, when $\lambda > 0$) is present. In comparison, we proved that the necessary regularity conditions are satisfied for poroelastic systems (i.e., $\delta = 0$) without Tikhonov term (i.e., $\lambda = 0$).

Table 4: Control function ψ, poroelastic case $\delta = 0$: Number of iterations required by the executions of Algorithm 1 for the different initializations of the computational example over the different values of ε of the homotopy, cumulative for the homotopy (H), and for the unregularized problem (U).

	Ini.	$\varepsilon =$ 0.016	0.008	0.004	0.002	0.001	0.0005	0.00025	0	(H)	(U)
$\lambda = 0$	1	30	4	2	1	2	1	1	2	43	35
$\lambda = 10^{-4}$	1	31	2	2	2	1	1	1	2	42	35
$\lambda = 10^{-2}$	1	28	8	4	2	1	1	1	2	47	31
$\lambda = 0$	2	53	13	5	3	2	1	1	3	81	64
$\lambda = 10^{-4}$	2	57	5	7	5	2	1	1	1	79	47
$\lambda = 10^{-2}$	2	35	4	7	3	2	1	1	3	56	33
$\lambda = 0$	3	66	2	3	2	1	1	1	2	78	33
$\lambda = 10^{-4}$	3	60	2	12	3	2	1	1	3	84	38
$\lambda = 10^{-2}$	3	47	7	6	1	2	1	1	2	67	36
$\lambda = 0$	4	29	8	2	2	3	1	1	2	48	22
$\lambda = 10^{-4}$	4	23	4	6	3	4	1	1	1	43	22
$\lambda = 10^{-2}$	4	22	2	3	3	2	1	1	1	35	26
$\lambda = 0$	5	41	2	2	6	1	1	1	2	56	21
$\lambda = 10^{-4}$	5	49	2	2	4	8	1	1	4	71	26
$\lambda = 10^{-2}$	5	41	2	3	3	1	1	1	3	55	27
$\lambda = 0$	6	79	2	2	2	1	1	1	2	90	50
$\lambda = 10^{-4}$	6	58	2	2	2	1	1	1	2	69	49
$\lambda = 10^{-2}$	6	50	6	6	1	2	1	1	3	70	39

We applied Algorithm 1 to instances of (P) for differently scaled Tikhonov terms, for both $\delta = 0$ and $\delta = 1$, and for the two different control inputs that were analyzed before. We observed that the presence of the Tikhonov term does not seem to negatively impact the practical performance of Algorithm 1, although Assumption 4.1 is always violated in this case. If Assumption 4.1 is violated due to the choice $\delta = 1$ (i.e., in the poroviscoelastic case), then the performance of Algorithm 1 and the quality of the final iterates it produces before the trust region collapses is degraded. This can be alleviated by executing a homotopy that drives the support parameter of a mollification of the control input into the PDE to zero over the course of the optimization. However, the execution of the homotopy comes at a higher computational cost.

Table 5: Control function ψ, poroviscoelastic case $\delta = 1$: Number of iterations required by the executions of Algorithm 1 for the different initializations of the computational example over the different values of ε of the homotopy, cumulative for the homotopy (H), and for the unregularized problem (U).

	$\varepsilon =$ Ini.	0.016	0.008	0.004	0.002	0.001	0.0005	0.00025	0	(H)	(U)
$\lambda = 0$	1	120	59	44	23	11	1	1	6	265	48
$\lambda = 10^{-4}$	1	112	40	35	26	8	1	1	9	232	40
$\lambda = 10^{-2}$	1	102	58	43	19	11	1	1	5	240	35
$\lambda = 0$	2	145	62	47	24	6	1	1	3	289	62
$\lambda = 10^{-4}$	2	134	64	38	15	17	1	1	8	278	57
$\lambda = 10^{-2}$	2	194	61	43	30	13	1	1	6	349	67
$\lambda = 0$	3	99	56	62	31	10	1	1	8	268	26
$\lambda = 10^{-4}$	3	117	45	50	18	14	1	1	16	262	31
$\lambda = 10^{-2}$	3	119	36	44	29	7	1	1	4	241	26
$\lambda = 0$	4	64	54	42	27	9	1	1	3	201	24
$\lambda = 10^{-4}$	4	62	75	34	17	12	1	1	5	207	22
$\lambda = 10^{-2}$	4	73	45	39	16	15	1	1	4	194	22
$\lambda = 0$	5	104	57	37	22	11	1	1	7	240	34
$\lambda = 10^{-4}$	5	132	50	37	26	7	1	1	10	264	34
$\lambda = 10^{-2}$	5	111	60	57	30	17	1	1	6	283	41
$\lambda = 0$	6	131	78	42	23	19	1	1	8	303	43
$\lambda = 10^{-4}$	6	145	73	43	23	19	1	1	7	312	43
$\lambda = 10^{-2}$	6	194	73	39	23	12	1	1	6	349	64

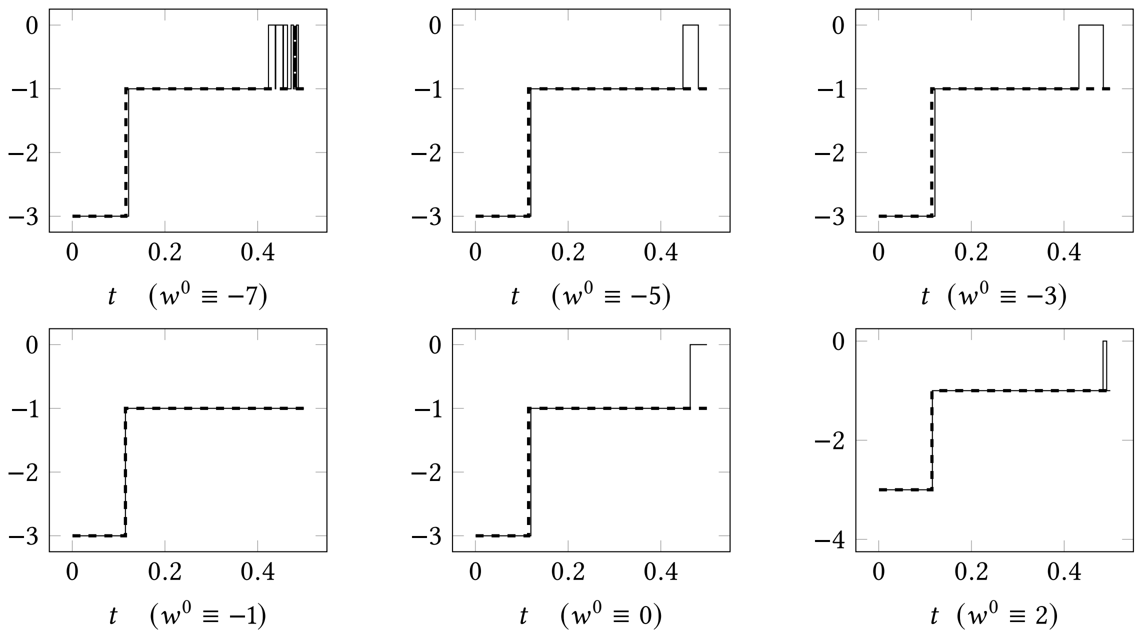

Figure 2: Control function ψ, poroviscoelastic case $\delta = 1$, $\lambda = 10^{-2}$: final control functions produced for the unregularized optimization (solid) and for the homotopy (dashed).

Table 6: Control function S, poroelastic case $\delta = 0$: Objective, instationarity, and (cumulative) outer iterations of (P) for six initial controls achieved by executing Algorithm 1 on the discretized problem as well as by executing a homotopy, abbreviated (H), of executions of Algorithm 1 on regularized problems with regularization parameter driven to zero for the same initial controls.

	Ini.	Final Obj.	Final Obj. (H)	Final Inst.	Final Inst. (H)	Iter.	Cum. Iter. (H)
$\lambda = 0$	1	1.2420	1.2417	1.0×10^{-5}	9.1×10^{-6}	16	24
$\lambda = 10^{-4}$	1	1.2420	1.2417	1.0×10^{-5}	9.2×10^{-6}	16	24
$\lambda = 10^{-2}$	1	1.2454	1.2451	2.0×10^{-5}	1.9×10^{-5}	16	24
$\lambda = 0$	2	1.2423	1.2421	1.5×10^{-5}	1.2×10^{-5}	25	33
$\lambda = 10^{-4}$	2	1.2423	1.2422	1.5×10^{-5}	1.2×10^{-5}	25	33
$\lambda = 10^{-2}$	2	1.2602	1.2455	4.4×10^{-7}	2.4×10^{-6}	29	40
$\lambda = 0$	3	1.2428	1.2424	8.8×10^{-6}	1.2×10^{-5}	24	32
$\lambda = 10^{-4}$	3	1.2429	1.2424	8.7×10^{-6}	1.3×10^{-5}	26	32
$\lambda = 10^{-2}$	3	1.2462	1.2458	9.5×10^{-7}	2.2×10^{-5}	26	32
$\lambda = 0$	4	1.2417	1.2428	1.1×10^{-5}	1.5×10^{-5}	15	27
$\lambda = 10^{-4}$	4	1.2417	1.2428	1.1×10^{-5}	1.5×10^{-5}	15	27
$\lambda = 10^{-2}$	4	1.2450	1.2462	2.1×10^{-5}	2.5×10^{-5}	15	29
$\lambda = 0$	5	1.2414	1.2423	1.1×10^{-5}	1.6×10^{-5}	12	23
$\lambda = 10^{-4}$	5	1.2414	1.2424	1.1×10^{-5}	1.6×10^{-5}	12	23
$\lambda = 10^{-2}$	5	1.2448	1.2457	2.1×10^{-5}	6.3×10^{-6}	12	24
$\lambda = 0$	6	1.2553	1.2556	1.1×10^{-5}	2.5×10^{-5}	16	28
$\lambda = 10^{-4}$	6	1.2554	1.2556	1.1×10^{-5}	2.5×10^{-5}	16	28
$\lambda = 10^{-2}$	6	1.2602	1.2605	1.1×10^{-5}	1.0×10^{-5}	13	29

Table 7: Control function S, poroviscoelastic case $\delta = 1$: Objective, instationarity, and (cumulative) outer iterations of (P) for six initial controls achieved by executing Algorithm 1 on the discretized problem as well as by executing a homotopy, abbreviated (H), of executions of Algorithm 1 on regularized problems with regularization parameter driven to zero for the same initial controls.

	Ini.	Final Obj.	Final Obj. (H)	Final Inst.	Final Inst. (H)	Iter.	Cum. Iter. (H)
$\lambda = 0$	1	1.8162	1.4783	3.6×10^{-3}	5.2×10^{-5}	10	45
$\lambda = 10^{-4}$	1	1.8162	1.4783	3.6×10^{-3}	5.2×10^{-5}	10	45
$\lambda = 10^{-2}$	1	1.8185	1.4814	3.5×10^{-3}	4.2×10^{-5}	10	42
$\lambda = 0$	2	1.5849	1.4930	1.2×10^{-3}	2.3×10^{-4}	20	57
$\lambda = 10^{-4}$	2	1.5849	1.4930	1.2×10^{-3}	2.3×10^{-4}	20	57
$\lambda = 10^{-2}$	2	1.5755	1.4971	1.0×10^{-3}	2.2×10^{-4}	21	58
$\lambda = 0$	3	1.5480	1.5060	9.7×10^{-4}	1.5×10^{-4}	17	53
$\lambda = 10^{-4}$	3	1.5481	1.5060	9.7×10^{-4}	1.5×10^{-4}	17	53
$\lambda = 10^{-2}$	3	1.6598	1.5092	1.5×10^{-3}	1.4×10^{-4}	17	58
$\lambda = 0$	4	2.0874	1.5176	3.9×10^{-3}	3.1×10^{-4}	15	49
$\lambda = 10^{-4}$	4	2.0874	1.5177	3.9×10^{-3}	3.1×10^{-4}	15	49
$\lambda = 10^{-2}$	4	2.0279	1.4846	3.6×10^{-3}	7.3×10^{-5}	18	50
$\lambda = 0$	5	1.7936	1.4807	1.0×10^{-3}	3.6×10^{-5}	8	46
$\lambda = 10^{-4}$	5	1.7936	1.4808	1.0×10^{-3}	3.6×10^{-5}	8	46
$\lambda = 10^{-2}$	5	1.8013	1.4838	1.2×10^{-3}	2.6×10^{-5}	10	49
$\lambda = 0$	6	1.4730	1.4713	3.6×10^{-4}	2.8×10^{-4}	14	45
$\lambda = 10^{-4}$	6	1.4731	1.4713	3.6×10^{-4}	2.8×10^{-4}	14	45
$\lambda = 10^{-2}$	6	1.4776	1.4752	3.6×10^{-4}	2.2×10^{-4}	14	46

Table 8: Control function S, poroelastic case $\delta = 0$: Number of iterations required by the executions of Algorithm 1 forthe different initializations of the computational example over the different values of ε of the homotopy, cumulative for the homotopy (H), and for the unregularized problem (U).

	Ini.	$\varepsilon =$ 0.016	0.008	0.004	0.002	0.001	0.0005	0.00025	0	(H)	(U)
$\lambda = 0$	1	14	2	1	1	1	1	2	2	24	16
$\lambda = 10^{-4}$	1	14	2	1	1	1	1	2	2	24	16
$\lambda = 10^{-2}$	1	14	2	1	1	1	1	2	2	24	16
$\lambda = 0$	2	20	1	1	2	2	3	1	3	33	25
$\lambda = 10^{-4}$	2	20	1	1	2	2	3	1	3	33	25
$\lambda = 10^{-2}$	2	25	3	1	3	2	2	1	3	40	29
$\lambda = 0$	3	19	3	1	2	3	1	1	2	32	24
$\lambda = 10^{-4}$	3	19	3	1	2	3	1	1	2	32	26
$\lambda = 10^{-2}$	3	19	2	2	2	3	1	1	2	32	26
$\lambda = 0$	4	16	2	1	2	3	1	1	1	27	15
$\lambda = 10^{-4}$	4	16	2	1	2	3	1	1	1	27	15
$\lambda = 10^{-2}$	4	16	3	1	3	3	1	1	1	29	15
$\lambda = 0$	5	11	1	2	3	2	2	1	1	23	12
$\lambda = 10^{-4}$	5	11	1	2	3	2	2	1	1	23	12
$\lambda = 10^{-2}$	5	12	1	2	2	2	3	1	1	24	12
$\lambda = 0$	6	15	2	2	1	4	2	1	1	28	16
$\lambda = 10^{-4}$	6	15	2	2	1	4	2	1	1	28	16
$\lambda = 10^{-2}$	6	12	2	3	1	3	1	4	3	29	13

Table 9: Control function S, poroviscoelastic case $\delta = 1$: Number of iterations required by the executions of Algorithm 1 for the different initializations of the computational example over the different values of ε of the homotopy, cumulative for the homotopy (H), and for the unregularized problem (U).

	Ini.	$\varepsilon =$ 0.016	0.008	0.004	0.002	0.001	0.0005	0.00025	0	(H)	(U)
$\lambda = 0$	1	21	4	9	2	1	3	3	2	45	10
$\lambda = 10^{-4}$	1	21	4	9	2	1	3	3	2	45	10
$\lambda = 10^{-2}$	1	20	10	2	1	1	3	3	2	42	10
$\lambda = 0$	2	36	4	4	3	3	3	2	2	57	20
$\lambda = 10^{-4}$	2	36	4	4	3	3	3	2	2	57	20
$\lambda = 10^{-2}$	2	38	4	3	3	3	3	2	2	58	21
$\lambda = 0$	3	25	16	2	2	2	1	3	2	53	17
$\lambda = 10^{-4}$	3	25	16	2	2	2	1	3	2	53	17
$\lambda = 10^{-2}$	3	29	17	2	2	2	1	3	2	58	17
$\lambda = 0$	4	31	4	3	3	1	2	3	2	49	15
$\lambda = 10^{-4}$	4	31	4	3	3	1	2	3	2	49	15
$\lambda = 10^{-2}$	4	30	4	3	3	1	2	3	4	50	18
$\lambda = 0$	5	31	2	1	3	3	2	2	2	46	8
$\lambda = 10^{-4}$	5	31	2	1	3	3	2	2	2	46	8
$\lambda = 10^{-2}$	5	31	2	1	3	3	3	4	2	49	10
$\lambda = 0$	6	26	3	2	2	5	3	2	2	45	14
$\lambda = 10^{-4}$	6	26	3	2	2	5	3	2	2	45	14
$\lambda = 10^{-2}$	6	27	3	2	2	5	3	2	2	46	14

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